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On graph orientation and the number of vertices in a longest directed path

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We use the same terminology as in [1]. Let G be a graph. $\lambda(G)$ is the number of vertices in a longest path in G. For a directed graph D, we define $\lambda(D)$ to be the number of vertices in a longest directed path in D. Suppose that k is an integer such that $\chi(G) \leq k \leq \lambda(G)$. In [1] C. Lin asked whether it is always possible to orient G such that the number of vertices in a longest directed path in the orientation of G is k. In this note we give an affirmative answer to that question. The background of this problem and the relation between the chromatic number of graph G and the path length in orientations of G can be found in [1].

Theorem 1 *For a graph G with $\chi(G) = t$ and $\lambda(G) = l$ and an integer k such that $t \leq k \leq l$, there always exists an orientation D of G such that $\lambda(D) = k$.*

Proof: Suppose that $P = v_1 v_2 \cdots v_l$ is a path of l vertices in G. Let C be a t-colouring of G such that the colour classes are $V_1, V_2, ..., V_t$ and $v_1 \in V_t$. We define a $(t+m-1)$-colouring of G recursively for $m = 1, 2, ..., l$. For $m = 1$ and $i = 1, 2, ..., t$ let $V_i^1 = V_i$. For $m = 2, 3, ..., l$, we define $V_i^m = V_i^{m-1} \setminus \{v_m\}$ for $i = 1, 2, ..., t+m-2$ and $V_{t+m-1}^m = \{v_m\}$. What this means is that at the mth step, we take the vertex v_m and colour it with a new colour while keep the colours of all the other vertices. Therefore, this is a proper colouring of G. Let D_m be the orientation of G induced by this $(t+m-1)$-colouring. That is, in D_m, the edge uv is directed from u to v if $u \in V_i^m$, $v \in V_j^m$ and $i < j$.
Claim: $\lambda(D_m) \leq \lambda(D_{m-1}) + 1$ for all $m = 2, ..., l$.

Let Q be a longest directed path in D_m. We prove this claim in two cases.

Case 1: Q does not contain the vertex v_m. Since all the edges that are not incident with v_m are directed the same way in D_m as in D_{m-1}, Q is also a directed path in D_{m-1} in this case. We have $\lambda(D_m) \leq \lambda(D_{m-1})$.

Case 2: Q contains the vertex v_m. Since all the edges that are incident with v_m are directed towards v_m in D_m, v_m must be the last vertex in Q. Therefore $Q \backslash \{v_m\}$ is a directed path in D_{m-1}. We have $\lambda(D_{m-1}) \geq \lambda(D_m) - 1$. This completes the proof of the claim.

In D_l, the edge $v_i v_{i+1}$ is directed from v_i to v_{i+1} for all $i = 1, 2, ..., l-1$. P is a directed path in D_l. Therefore $\lambda(D_l) = l$. Since $\lambda(D_1) = t$ and $t \leq k \leq l$, by the claim, there must be a j such that $\lambda(D_j) = k$. □

An efficient algorithm can easily be derived to find the orientation in the theorem when a longest path and a t-colouring of G is given.

References

1. C. Lin: Simple proofs of results on paths representing all colors in proper vertex-colorings. Graphs and Combinatorics **23**, 201-203 (2007).