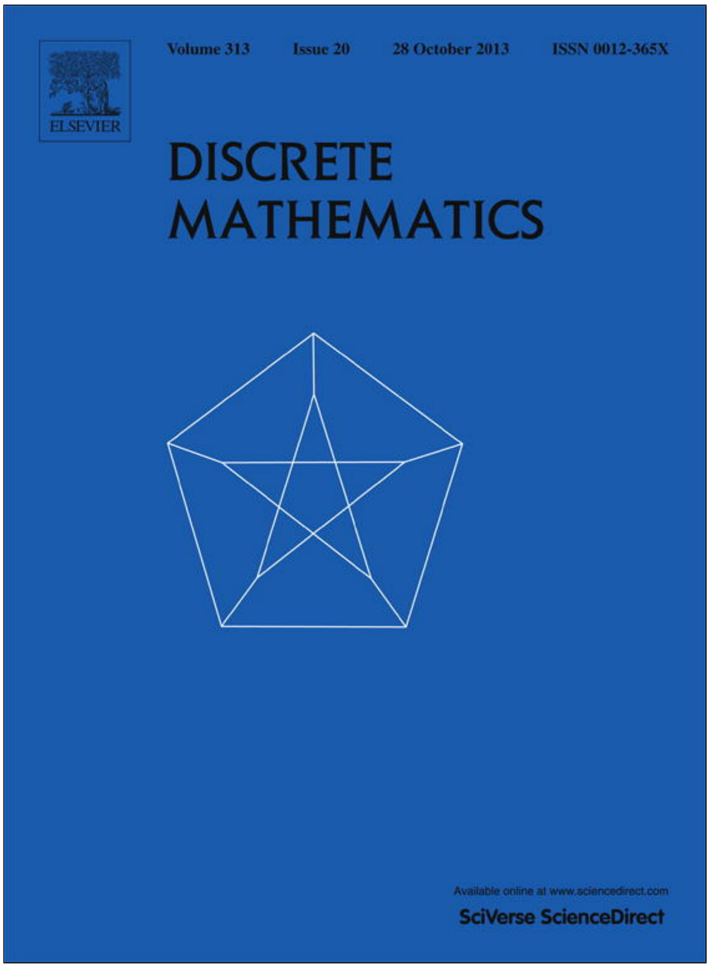
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Note

A lower bound for the chromatic capacity in terms of the chromatic number of a graph

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ABSTRACT

When the vertices and edges are coloured with k colours, an edge is called monochromatic if the edge and the two vertices incident with it all have the same colour. The chromatic capacity of a graph G, $\chi_{CAP}(G)$, is the largest integer k such that the edges of G can be coloured with k colours in such a way that when the vertices of G are coloured with the same set of colours, there is always a monochromatic edge. It is easy to see that $\chi_{CAP}(G) \le \chi(G) - 1$. Greene has conjectured that there is an unbounded function f such that $\chi_{CAP}(G) \ge f(\chi(G))$. In this article we prove Greene's conjecture.

1. Introduction

Consider a 2-player game played on a graph G with k colours. Player A colours the edges of G with these k colours. When A finishes, player B uses the same set of colours to colour the vertices of G. An edge e is monochromatic if e and both vertices incident with e are all the same colour. B wins the game if he can colour the vertices of G without creating any monochromatic edges; otherwise A wins. The largest number of colours for which the player A has a winning strategy is the chromatic capacity of G, denoted by $\chi_{CAP}(G)$. Therefore, if $\chi_{CAP}(G) = k$, the edges of G can be coloured with k colours such that for every colouring of the vertices of G using the same set of colours, there is at least one monochromatic edge; and if $k+1$ colours are allowed, for each edge-colouring of G, the vertices of G can be coloured such that there is no monochromatic edges. The term colour capacity was first used by Archer in [1]. The same concept was also used in [3,4] and it was applied mostly to complete graphs (the term used there was split colouring).

If the vertices of G can be coloured properly with k colours, then player B will win no matter how the edges of G are coloured. Thus we have the bound $\chi_{CAP}(G) \le \chi(G) - 1$. There are many examples showing that this bound is tight. Huizenga [6] constructed graphs G with the property $\chi_{CAP}(G) = \chi(G) - 1$ and no odd cycle of length less than q for any positive integer q. He also asked whether it is possible to construct a graph G with the property $\chi_{CAP}(G) = \chi(G) - 1$ and without any cycle of length less than q for any given positive integer q. This problem was solved in [7] with a description of a method to construct such graphs.

In contrast to the upper bound, the known results about the lower bound of $\chi_{CAP}(G)$ in terms of $\chi(G)$ are less satisfactory. It was proved by several authors independently in [2,3] that $\chi_{CAP}(K_n)$ is in the order of $\sqrt{n}$. Since $\chi(K_n) = n$, we have

$$\chi_{CAP}(G) = \Theta\left(\sqrt{\chi(G)}\right)$$

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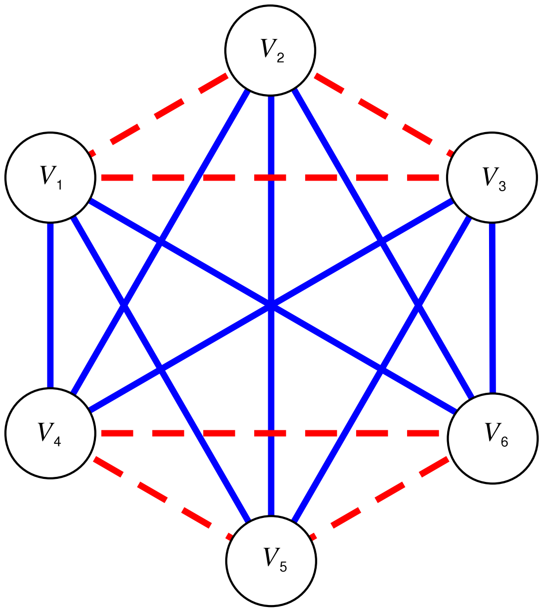

Fig. 1. Edge-colouring of a 6-chromatic graph G. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

if G is a complete graph. For a general graph G, Greene in [5] showed that

$$\chi_{CAP}(G)^2 \ln \chi_{CAP}(G) > (1 - o(1)) \frac{\chi(G)^2}{2n}$$

where n is the number of vertices in G. Greene conjectured:

Conjecture 1 (*Greene's Conjecture*)**.** *There exists an unbounded function* $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ *such that* $\chi_{CAP}(G) > f(\chi(G))$ *holds.*

Greene's conjecture was regarded as possibly the most important open question in the study of chromatic capacities [6]. Also in [6], Huizenga proved that Green's conjecture is true for almost all graphs. The goal of this note is to prove Green's conjecture for all graphs.

2. The lower bound

Our main result is Theorem 5, in which we obtain a lower bound of $\chi_{CAP}(G)$ that depends on $\chi(G)$ only, thus settling Green's Conjecture in the affirmative. In [5], Greene characterised all graphs G with $\chi_{CAP}(G) = 1$ and obtained the bound that if $\chi(G) \geq 4$ then $\chi_{CAP}(G) \geq 2$. This method cannot be easily applied in the general case. Our approach is to show that when $\chi(G)$ is large then A has a winning strategy with a smaller but still relatively large number of colours; thus $\chi_{CAP}(G)$ is relatively large. To demonstrate our method, we prove a weaker result in Theorem 3 for the case $\chi_{CAP}(G) \geq 2$. That is, we show that $\chi(G) \geq 6$ implies $\chi_{CAP}(G) \geq 2$. The method is then generalised to prove Theorem 5.

Lemma 2. *Let G be a graph with $\chi(G) \leq 6$. There is an 2-edge colouring of G such that if the vertices of an induced subgraph H of G can be 2-coloured with no monochromatic edges, then (the vertices of) H can be properly coloured with five colours.*

Proof. Let the colour classes of G be $V_1, V_2, \ldots, V_6$ (some of them might which could be empty). Let $G_1 = V_1 \cup V_2 \cup V_3$ and $G_2 = V_4 \cup V_5 \cup V_6$. We colour the edges of G with colours 1 and 2 as follows: An edge xy will be coloured with colour 1 if both x and y are in G_1 or both x and y are in G_2. Otherwise, it is coloured with colour 2.

The thick lines in Fig. 1 connecting V_i and V_j represent all the edges in G that have one end in V_i and another end in V_j. The edges represented by the broken (red) lines are coloured with colour 1 and the edges represented by the solid (blue) lines are coloured with colour 2.

Suppose that the vertices of an induced subgraph H of G are then coloured with colours 1 and 2 such that there is no monochromatic edge. Let $W_i = V_i \cap V(H)$. Let $W_{i,1}$ be the set of vertices in W_i that are coloured with colour 1 and $W_{i,2}$ be the set of vertices in W_i that are coloured with colour 2, for $i = 1, 2, \ldots, 6$. We have $W_i = W_{i,1} \cup W_{i,2}$ for all i. $W_{i,2} \cup W_{i+3,2}$ are independent sets for $i = 1, 2, 3$. This is because each $W_{j,2}$ is independent for all j, and if there is an edge xy such that $x \in W_{i,2}$ and $y \in W_{i+3,2}$ then the edge is coloured with colour 2 and both vertices are coloured with colour 2, a contradiction. Similarly, each of $W_{1,1} \cup W_{2,1} \cup W_{3,1}$ and $W_{4,1} \cup W_{5,1} \cup W_{6,1}$ is an independent set. These five independent sets induce a 5-colouring of H as illustrated in Fig. 2. □

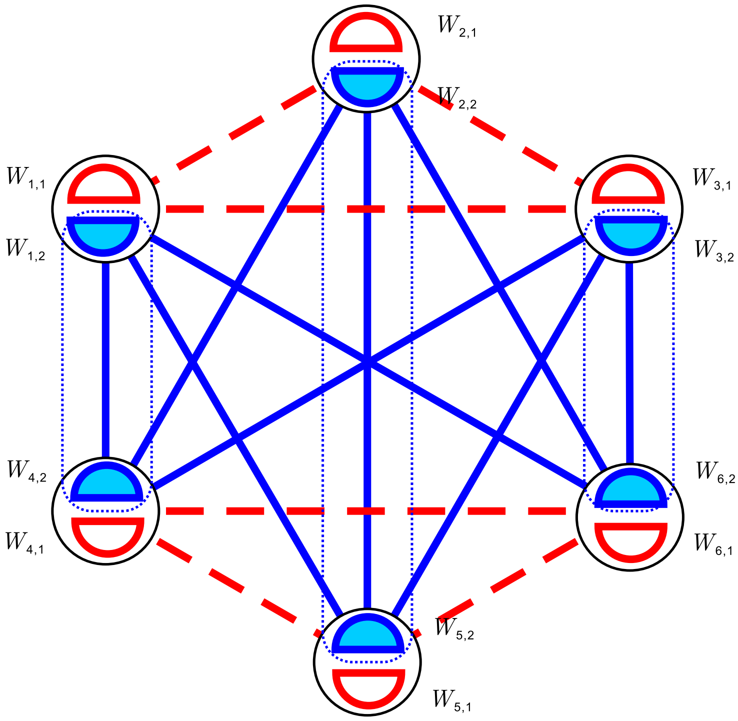

Fig. 2. A 5-colouring of H.

Theorem 3. *If* $\chi(G) = 6$ *then* $\chi_{CAP}(G) \geq 2$.

Proof. By letting H be the graph G itself and applying Lemma 2, it follows that if $\chi(G) = 6$ player A has a winning strategy with 2 colours. Therefore $\chi_{CAP}(G) \geq 2$. □

The subgraph H is not necessary to prove the result of Theorem 3, but it is needed in the general case for the inductive step.

Lemma 4. *Let* $f : \mathbb{N} \to \mathbb{N}$ *be a function such that* $f(2) = 6$ *and* $f(k) \geq 2 \cdot [f(k-1)]^2$ *for all* $k \geq 3$. *Suppose that* G *is a graph such that* $\chi(G) \leq f(k)$ *for some integer* $k \geq 2$. *There exists a* k*-edge colouring of* G *such that if the vertices of an induced subgraph* H *of* G *can be* k*-coloured with no monochromatic edges, then the vertices of* H *can be properly coloured with* $f(k) - 1$ *colours.*

Proof. We prove the statement by induction on k. The base step $k = 2$ is covered in Lemma 2. For the inductive step, we assume that $k \geq 3$ and the theorem is true for $k - 1$. Let $f(k-1) = r, f(k) = t$ and $\left\lceil \frac{t}{r} \right\rceil = m$. $t \geq 2r^2$ implies $m \geq 2r$. Let $V_1, V_2, \ldots, V_t$ be the colour classes of a t-colouring of G. Let $G_1, G_2, \ldots, G_m$ be the induced subgraphs of G such that

$$V(G_1) = V_1 \cup V_2 \cup \cdots \cup V_r$$
$$V(G_2) = V_{r+1} \cup V_{r+2} \cup \cdots \cup V_{2r}$$
$$\vdots$$
$$V(G_m) = V_{(m-1)r+1} \cup V_{(m-1)r+2} \cup \cdots \cup V_t.$$

We have $\chi(G_i) \leq r$ for $i = 1, 2, \ldots, m$. The edges of G will be coloured with colour set $\{1, 2, \ldots, k\}$ as follows: edges inside a subgraph G_i will be coloured using colours $\{1, 2, \ldots, k-1\}$ according to the $(k-1)$-edge colouring provided by the inductive hypothesis. Edges with one end in G_i and another end in G_j for some $i \neq j$ will be coloured with colour k. For this edge colouring, suppose that there is a subgraph H of G and a vertex colouring $\mathcal{C}$ using colours $\{1, 2, \ldots, k\}$ such that there is no monochromatic edge in H. Let $H_i = H \cap G_i$ $(i = 1, 2, \ldots, m)$ and $W_j = V(H) \cap V_j$ for $j = 1, 2, \ldots, t$. Let $W_{i,k}$ be the vertices in W_i that are coloured with colour k in $\mathcal{C}$. Each one of

$$W_{1,k} \cup W_{r+1,k} \cup \cdots \cup W_{(m-1)r+1,k},$$
$$W_{2,k} \cup W_{r+2,k} \cup \cdots \cup W_{(m-1)r+2,k},$$
$$\vdots$$
$$W_{r,k} \cup W_{2r,k} \cup \cdots \cup W_{\lfloor \frac{t}{r} \rfloor r,k}$$

is an independent set. The subgraphs $H_1 \setminus \left\{W_{1,k} \cup W_{2,k} \cup \cdots \cup W_{r,k}\right\}$, $H_2 \setminus \left\{W_{r+1,k} \cup W_{r+12,k} \cup \cdots \cup W_{2r,k}\right\}, \ldots, H_m \setminus \left\{W_{(m-1)r+1,k} \cup W_{(m-1)r+2,k} \cup \cdots \cup W_{\lfloor \frac{t}{r} \rfloor r,k}\right\}$ are coloured with colours $\{1, 2, \ldots, k-1\}$ without monochromatic edges. By the inductive hypothesis, each of them can be partitioned into $r-1$ independent sets. Therefore, the vertex set of H can be partitioned into $r + m(r-1)$ independent sets. Since $m \geq 2r$ and $t \geq (m-1)r + 1$, we have

$$\begin{aligned} r + m(r-1) &= mr + r - m \\ &\leq (m+1)r - 2r = (m-1)r \\ &\leq t - 1. \end{aligned}$$

So H can be coloured with $f(k) - 1$ colours. □

It is easy to verify that $f(k) = C \cdot 2^{2^{2k}}$ satisfies the condition $f(k) \geq 2 \cdot [f(k-1)]^2$ for every positive constant C. We choose $C = \frac{3}{2^{15}}$ so that $f(2) = 6$ giving us the main result.

Theorem 5. *If $\chi(G) = \frac{3}{2^{15}} \cdot 2^{2^{2k}}$, then $\chi_{CAP}(G) \geq k$.*

Proof. The edge colouring with colours $\{1, 2, \ldots, k\}$ in the proof of Lemma 4 is a winning strategy for Player A. Let $f(k) = \frac{3}{2^{15}} \cdot 2^{2^{2k}}$. If Player B can colour the vertices of G with no monochromatic edge, then according to Lemma 4, G can be coloured properly with $\frac{3}{2^{15}} \cdot 2^{2^{2k}} - 1$ colours, a contradiction. □

From Theorem 5, we have

Corollary 6.

$$\chi_{CAP}(G) \geq K \log\log(\chi(G))$$

for some constant K and all graphs G. Thus a lower bound of $\chi_{CAP}(G)$ is $\Theta(\log\log(\chi(G)))$.

As far as we know, there are no known graphs with chromatic capacity smaller than the order of $\sqrt{\chi(G)}$. The bound in Corollary 6 is likely not the best possible. It would be interesting to know whether there are any families of graphs whose chromatic capacities are smaller than the order of $\sqrt{\chi(G)}$, perhaps of the order of $\log(\chi(G))$.

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