Math 306 —Complex Analysis

Final Exam; April 19, 2004 —Solutions

1. Let $f : \mathbb{C}\longrightarrow\mathbb{C}$ be the complex-inversion map, ie. $f(z) = \frac{1}{z}$.

$(\frac{8}{200})$ (a) Suppose $z = x + \mathbf{i}y$ and $f(z) = u(x,y) + v(x,y)\mathbf{i}$ for some functions $u, v : \mathbb{R}^2\longrightarrow\mathbb{R}$. Express $u(x,y)$ and $v(x,y)$ in terms of x and y.

Solution: Suppose $z = x + \mathbf{i}y$. If $z = r\angle\theta$, where $r = \sqrt{x^2+y^2}$, then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Thus,

$$\begin{aligned} f(z) &= \frac{1}{r}\angle(-\theta) = \frac{1}{r}\Big(\cos(-\theta)+\mathbf{i}\sin(-\theta)\Big) = \frac{1}{r^2}\cdot r\Big(\cos(\theta)-\mathbf{i}\sin(\theta)\Big) \\ &= \frac{1}{r^2}(x-\mathbf{i}y) = \left(\frac{x}{x^2+y^2}\right) - \left(\frac{y}{x^2+y^2}\right)\mathbf{i} \end{aligned}$$

Hence, $\boxed{u(x,y) = \frac{x}{x^2+y^2}}$ and $\boxed{v(x,y) = \frac{-y}{x^2+y^2}}$.

$(\frac{7}{200})$ (b) Show that f satisfies the Cauchy-Riemann equations everywhere except at $z = 0$. Hence f is analytic everywhere except at the origin.

Solution: First observe that

$$\begin{aligned} \partial_x u(x,y) &= \frac{(x^2+y^2)-x\cdot(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \partial_y u(x,y) &= \frac{-x\cdot(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}. \\ \partial_x v(x,y) &= \frac{y\cdot(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}. \\ \partial_y v(x,y) &= \frac{-(x^2+y^2)+y\cdot(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}$$

It follows that $\partial_x u(x,y) = \partial_y v(x,y)$ and $\partial_y u(x,y) = -\partial_x v(x,y)$, as desired.

(c) Let $z = r\angle\theta$, and let $z_1 = r_1\angle\theta$ be another nearby point with the same angle, but with $r_1 < r$.

$(\frac{5}{200})$ i. Express $z - z_1$ in polar coordinates.
Then express $z_1 - z$ in polar coordinates. [**Hint:** *Drawing a picture may help.*]

Solution: $z - z_1 = (r\angle\theta) - (r_1\angle\theta) = (r-r_1)\angle\theta$. Note that $(r-r_1) > 0$ because $r_1 < r$.
Thus, $(z_1 - z) = -(z-z_1) = -1\cdot((r-r_1)\angle\theta) = \boxed{(r-r_1)\angle(\pi\pm\theta)}$.

$(\frac{5}{200})$ ii. Compute $f(z)$ and $f(z_1)$.

Solution: $f(z) = \boxed{\frac{1}{r}\angle(-\theta)}$ and $f(z_1) = \boxed{\frac{1}{r_1}\angle(-\theta)}$.

$(\frac{5}{200})$

iii. Express $f(z_1) - f(z)$ in polar coordinates

Solution: $f(z_1) - f(z) = \left(\frac{1}{r_1}\angle(-\theta)\right) - \left(\frac{1}{r}\angle(-\theta)\right) = \left(\frac{1}{r_1} - \frac{1}{r}\right)\angle(-\theta) = \boxed{\left(\frac{r - r_1}{r_1 \cdot r}\right)\angle(-\theta)}$ (because $r_1 < r$, so $\frac{1}{r_1} > \frac{1}{r}$).

$(\frac{5}{200})$

iv. Compute $\dfrac{f(z_1) - f(z)}{z_1 - z}$.

Solution: $$\frac{f(z_1) - f(z)}{z_1 - z} = \frac{\left(\frac{r-r_1}{r_1 \cdot r}\right)\angle(-\theta)}{(r - r_1)\angle(\theta \pm \pi)} = \boxed{\left(\frac{1}{r_1 \cdot r}\right)\angle(-2\theta \mp \pi).}$$

$(\frac{1}{200})$

v. Take the limit as $z_1 \to z$ (ie. as $r_1 \to r$) of $\dfrac{f(z_1) - f(z)}{z_1 - z}$.

Solution: $$\lim_{z_1 \to z} \frac{f(z_1) - f(z)}{z_1 - z} = \lim_{r_1 \to z} \left(\frac{1}{r_1 \cdot r}\right)\angle(-2\theta \mp \pi) = \frac{1}{r^2}\angle(-2\theta \mp \pi) = \boxed{\frac{-1}{z^2}.}$$

$(\frac{14}{200})$

vi. Use the answer to question **(v)** to deduce the value of $f'(z)$. Explain *carefully* why your reasoning is justified.

Solution: In (a) we showed that f is analytic at z. Thus, $f'(z)$ exists, and will be equal to $\lim_{z_1 \to z} \dfrac{f(z_1) - f(z)}{z_1 - z}$ if z_1 approaches z *from any direction*. Hence, $f'(z) = \boxed{\frac{1}{z^2}.}$

$(\frac{10}{200})$

(d) Recall that $f' = \partial_x u + \mathbf{i}\partial_x v$; use this to compute $f'(z)$, thereby confirming your answer to question **(c)vi**.

Solution:

$$f'(z) = \partial_x u + \mathbf{i}\partial_x v = \left(\frac{y^2 - x^2}{(x^2 + y^2)^2}\right) + \mathbf{i} \cdot \left(\frac{2xy}{(x^2 + y^2)^2}\right) = \frac{y^2 - x^2 + 2\mathbf{i}xy}{(x^2 + y^2)^2}$$
$$= \frac{-(x - \mathbf{i}y)^2}{(x^2 + y^2)^2} = -\left(\frac{x - \mathbf{i}y}{x^2 + y^2}\right)^2 = -\left(\frac{1}{z}\right)^2 = \boxed{\frac{-1}{z^2}.}$$

$(\frac{20}{200})$

2. Let $f, g : \mathbb{C} \longrightarrow \mathbb{C}$ be analytic functions. Let $\mathbf{U} \subset \mathbb{C}$ be an open subset, bounded by a simple closed curve $\Gamma \subset \mathbb{C}$. Suppose $f(z) = g(z)$ for all $z \in \Gamma$. Use the *Maximum Modulus Principle* to prove that $f(u) = g(u)$ for all $u \in \mathbf{U}$.

Solution: Let $h(z) = f(z) - g(z)$. Then h is an analytic function, so the Maximum Modulus Principle says that $|h(u)|$ takes its maximal value in $\mathbf{U}$ on the boundary curve Γ.

However, for all $z \in \Gamma$, we have $f(z) = g(z)$; hence $h(z) = f(z) - g(z) = 0$. Hence $|h(z)| = 0$ for all $z \in \Gamma$; since this is the maximum value of $|h(z)|$ in $\mathbf{U}$, we conclude that $|h(u)| = 0$ for all $u \in \mathbf{U}$. This means that $f(u) = g(u)$ for all $u \in \mathbf{U}$.

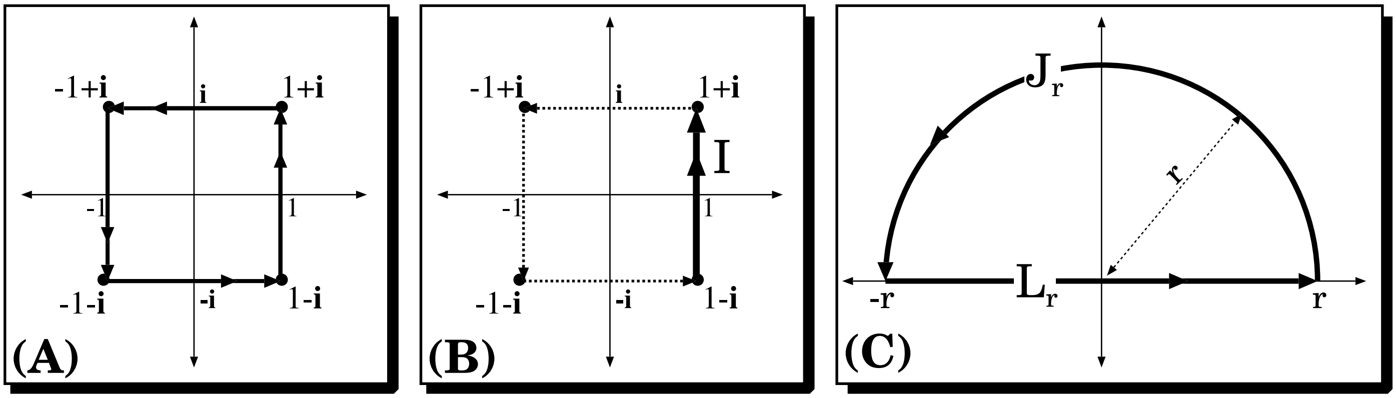

Figure 1: **(A)** $\square \subset \mathbb{C}$ is the square loop around zero with vertices at $(1+\mathbf{i})$, $(1-\mathbf{i})$, $(-1+\mathbf{i})$, and $(-1-\mathbf{i})$. [Question 4] **(B)** $\square = \sqsubset \cup \mathbf{I}$, where $\mathbf{I}$ is the right side of the box (ie. the line from $1-\mathbf{i}$ to $1+\mathbf{i}$) [Question 4c] **(C)** $\mathbf{K}_r = \mathbf{L}_r + \mathbf{J}_r$ [Question 5e]

(BONUS) 3. Suppose that $p(z)$ and $q(z)$ are polynomials, and that q has a simple root at $c \in \mathbb{C}$ (ie. $q(c) = 0$, but $q'(c) \neq 0$).

Thus, if $f(z) = \dfrac{p(z)}{q(z)}$, then $f(z)$ has a simple pole at c. Show that

$$\text{Residue}\,(f;\ c) \quad = \quad \frac{p(c)}{q'(c)}$$

Solution: Let $g(z) = f(z)\cdot(z-c)$. Then g is analytic near c, and the residue of $f(z)$ at c is just the value of $g(c)$. But

$$g(c) \;=\; \lim_{z\to c} g(z) \;=\; \lim_{z\to c}\,(z-c)\cdot\frac{p(z)}{q(z)} \;=\; \lim_{z\to c}\frac{p(z)}{\left(\dfrac{q(z)}{z-c}\right)}$$

$$= \frac{\lim\limits_{z\to c} p(z)}{\lim\limits_{z\to c}\left(\dfrac{q(z)}{z-c}\right)} \;=\; \frac{p(c)}{q'(c)}.$$

4. Let $f(z) = \frac{1}{z}$. Let $\square \subset \mathbb{C}$ be the square loop around zero with vertices at $(1+\mathbf{i})$, $(1-\mathbf{i})$, $(-1+\mathbf{i})$, and $(-1-\mathbf{i})$, as shown in Figure 1(A).

$(\frac{5}{200})$ (a) Does f have any poles inside the loop $\square$? If so, find the residue(s) of f at this/these pole(s).

Solution: Yes. f has $\boxed{\text{a pole at zero.}}$

Write $f(z) = \frac{1}{q(z)}$, where $q(z) = z$. Then Question 3 says that

$$\text{Residue}\,(f;0) \;=\; \frac{1}{q'(0)} \;=\; \frac{1}{1} \;=\; \boxed{1.}$$

$(\frac{5}{200})$

(b) Use Cauchy's Residue Formula to compute $\oint_{\Box} f(z)\ dz$

Solution: Cauchy's Residue Formula says that $\oint_{\Box} f(z)\ dz \;=\; 2\pi\mathbf{i}\cdot \text{Residue}\,(f;0) \;=$ $\boxed{2\pi\mathbf{i}.}$

$(\frac{15}{200})$

(c) Let $\Box = \sqsubset \cup \mathbf{I}$, where $\mathbf{I}$ is the right side of the box (ie. the line from $1-\mathbf{i}$ to $1+\mathbf{i}$) and "$\sqsubset$" is the other three sides of the box. [Figure 1(B)].

Use *parametric integration* to evaluate the path integral $\int_{\mathbf{I}} f(z)\ dz$.

Solution: Paramaterize $\mathbf{I}$ by $\gamma : [-1,1]\longrightarrow\mathbb{C}$ defined $\gamma(t) \;=\; 1+\mathbf{i}t$. Then $\dot{\gamma}(t) \;=\; \mathbf{i}$, and

$$\begin{aligned}
\int_{\mathbf{I}} f(z)\ dz \;&=\; \int_{-1}^{1} f\Big(\gamma(z)\Big)\,\dot{\gamma}(t)\ dt \;=\; \int_{-1}^{1} \frac{\mathbf{i}}{1+\mathbf{i}t}\ dt \;=\; \mathbf{i}\cdot\int_{-1}^{1} \frac{1-\mathbf{i}t}{\|1+\mathbf{i}t\|^2}\ dt \\
&=\; \mathbf{i}\cdot\int_{-1}^{1} \frac{1-\mathbf{i}t}{1+t^2}\ dt \;=\; \mathbf{i}\cdot\int_{-1}^{1} \frac{1}{1+t^2}\ dt \;+\; \int_{-1}^{1} \frac{t}{1+t^2}\ dt \\
&=\; \mathbf{i}\cdot\arctan(x)_{-1}^{1} \;+\; \frac{1}{2}\log(1+t^2)_{-1}^{1} \;=\; \mathbf{i}\cdot\left(\frac{\pi}{4}-\frac{-\pi}{4}\right) \;+\; \frac{1}{2}\Big(\log(2)-\log(2)\Big) \\
&=\; \boxed{\frac{\pi\mathbf{i}}{2}.}
\end{aligned}$$

$(\frac{5}{200})$

(d) Assume the path integral $\int f(z)\ dz$ on *each* of the other three sides of $\Box$ is *equal* to $\int_{\mathbf{I}} f(z)\ dz$. Use this to compute the value of $\oint_{\Box} f(z)\ dz$.

Solution: $\oint_{\Box} f(z)\ dz \;=\; 4\times\int_{\mathbf{I}} f(z)\ dz \;=\; 4\times\frac{\pi\mathbf{i}}{2}. \;=\; \boxed{2\pi\mathbf{i}.}$

5. Let n be an even number. Let $F_n(z) \;=\; \dfrac{1}{1+z^n}$.

$(\frac{10}{200})$

(a) Let ω be any nth root of (-1). Show that the *poles* of F_n are the elements of the set

$$\mathcal{P} \;=\; \left\{1\angle\left(\frac{k\pi}{n}\right)\ ;\ k \text{ any odd number}\right\}.$$

Solution: Let $z = r\angle\theta$. Then

$$\begin{aligned}
\Big(\ z \text{ is a pole of } F_n\ \Big) \;&\Longleftrightarrow\; \Big(\ 1+z^n=0\ \Big) \Longleftrightarrow \Big(\ z^n=-1\ \Big) \Longleftrightarrow \Big(\ r^n\angle(n\theta) \;=\; 1\angle\pi\ \Big) \\
&\Longleftrightarrow\; \Big(\ r=1 \text{ and } n\theta = k\pi \text{ for some odd } k\ \Big) \\
&\Longleftrightarrow\; \Big(\ r=1 \text{ and } \theta = k\pi/n \text{ for some odd } k\ \Big).
\end{aligned}$$

$(\frac{10}{200})$

(b) Let $\zeta = e^{\pi \mathbf{i}/n}$ [ie. $\zeta = 1\angle(\pi/n)$]. Let $\mathcal{Q}$ be the set of all poles of F_n in the upper half plane. Show that

$$\mathcal{Q} \quad = \quad \left\{\zeta^{2j+1} \; ; \; 0 \le j \le \frac{n}{2} - 1\right\}.$$

Solution: First note that $\mathcal{P} = \left\{1\angle\left(\frac{k\pi}{n}\right) \; ; \; k \text{ odd}\right\}. = \left\{1\angle\left(\frac{(2j+1)\pi}{n}\right) \; ; \; j \in \mathbb{N}\right\}.$

Now let $p \in \mathcal{P}$, and suppose $p = 1\angle\left(\frac{(2j+1)\pi}{n}\right)$. Then

$$\left(\ p \text{ is in the upper half plane}\ \right) \iff \left(\frac{(2j+1)\pi}{n} < \pi\right) \iff \left(\ 2j+1 < n\ \right)$$
$$\iff \left(\ j < \frac{n-1}{2}\ \right) \iff \left(\ j \le \frac{n-2}{2} = \frac{n}{2} - 1\ \right).$$

$(\frac{10}{200})$

(c) If $p \in \mathcal{P}$, show that $\text{Residue}\,(F_n;\ p) \;=\; \frac{-p}{n}$ [**Hint:** *Use question #3*]

Solution: Write $F_n(z) = \frac{1}{q(z)}$, where $q(z) = 1 + z^n$. Then $q'(z) = n \cdot z^{n-1}$, and question #3 says that

$$\text{Residue}\,(F_n;\ p) \quad = \quad \frac{1}{q'(p)} \quad = \quad \frac{1}{n \cdot p^{n-1}}$$

By hypothesis, $p^n = (-1)$. Thus, $p^{n-1} = \frac{-1}{p}$. Thus, $\frac{1}{n \cdot p^{n-1}} = \frac{-p}{n}$.

$(\frac{5}{200})$

(d) Show that $\sum_{q \in \mathcal{Q}} \text{Residue}\,(F_n; q) \quad = \quad \sum_{j=0}^{\frac{n}{2}-1} \frac{-\zeta^{2j+1}}{n}$

Solution: $\sum_{q \in \mathcal{Q}} \text{Residue}\,(F_n; q) \underset{(5c)}{=\!=} \sum_{q \in \mathcal{Q}} \frac{-q}{n} \underset{(5b)}{=\!=} \sum_{j=0}^{\frac{n}{2}-1} \frac{-\zeta^{2j+1}}{n}$

Here, (5b) is by 5b, (5c) is by 5c.

$(\frac{10}{200})$

(e) It can be shown that $\sum_{q \in \mathcal{Q}} \text{Residue}\,(F_n; q) \;=\; \frac{1}{\mathbf{i} n \sin(\frac{\pi}{n})}$. Let $\mathbf{K}_r = \mathbf{L}_r + \mathbf{J}_r$ be the curve in Figure 1(C) (assume $r > 1$). Compute the path integral

$$\oint_{\mathbf{K}_r} F_n(z)\ dz.$$

Solution: $\oint_{\mathbf{K}_r} F_n(z) \underset{(*)}{=\!=} 2\pi\mathbf{i} \cdot \sum_{q \in \mathcal{Q}} \text{Residue}\,(F_n; q) \underset{(5d)}{=\!=} 2\pi\mathbf{i} \cdot \left(\frac{1}{\mathbf{i} n \sin(\frac{\pi}{n})}\right)$

$$= \boxed{\frac{2\pi}{n \sin(\frac{\pi}{n})}.}$$

Here $(*)$ is by Cauchy's Residue Theorem, and (5d) is by by part 5d.

$(\frac{10}{200})$

(f) Prove that $\left|\int_{\mathbf{J}_r} F_n(z)\right| < \frac{\pi r}{r^n - 1}$.

Solution: $\left|\int_{\mathbf{J}_r} F_n(z)\right| \leq \text{length}\,[\mathbf{J}_r] \cdot \max_{z\in\mathbf{J}_r} |F_n(z)| = (\pi \cdot r) \cdot \max_{z\in\mathbf{J}_r} \left|\frac{1}{1+z^n}\right|$.

$$= \frac{\pi r}{\min_{z\in\mathbf{J}_r} |1+z^n|} \underset{\triangle}{=} \frac{\pi r}{r^n - 1}.$$

Here, $\triangle$ is the Triangle Inequality.

$(\frac{15}{200})$

(g) Now let $r\to\infty$, and compute $\int_{-\infty}^{\infty} \frac{1}{1+x^n}\, dx$.

Solution: First note that part 5f implies that

$$\lim_{r\to\infty} \left|\int_{\mathbf{J}_r} F_n(z)\, dz\right| \leq \lim_{r\to\infty} \frac{\pi r}{r^n - 1} = 0. \tag{1}$$

Also,

$$\frac{2\pi}{n\sin(\frac{\pi}{n})} \underset{(\diamond)}{=} \lim_{r\to\infty} \oint_{\mathbf{K}_r} F_n(z)\, dz = \lim_{r\to\infty} \int_{\mathbf{J}_r} F_n(z)\, dz + \lim_{r\to\infty} \int_{\mathbf{L}_r} F_n(z)\, dz$$
$$= \lim_{r\to\infty} \int_{\mathbf{J}_r} F_n(z)\, dz + \lim_{r\to\infty} \int_{-r}^{r} \frac{1}{1+x^n}\, dx.$$
$$\underset{(\dagger)}{=} \lim_{r\to\infty} \int_{-r}^{r} \frac{1}{1+x^n}\, dx = \int_{-} \infty^{\infty} \frac{1}{1+x^n}\, dx.$$

Here, $(\diamond)$ is by 5e, and $(\dagger)$ is by eqn.(1).

We conclude that $\int_{-\infty}^{\infty} \frac{1}{1+x^n}\, dx = \boxed{\frac{2\pi}{n\sin(\frac{\pi}{n})}}.$

6. Figure 2 shows a 'cross section' of the complex plane $\mathbb{C}$ and the Riemann sphere $\widehat{\mathbb{C}}$. In this picture, z is a point in $\mathbb{C}$, and $\mathbf{L}$ is a straight line connecting z to the north pole N of the sphere $\widehat{\mathbb{C}}$. The line $\mathbf{L}$ intersects the sphere $\widehat{\mathbb{C}}$ at the point ζ. If you drop a vertical line $\mathbf{V}$ straight down from ζ, then $\mathbf{V}$ intersects $\widehat{\mathbb{C}}$ again at Ω. If you draw a line from $\widehat{w}$ to N, then this line intersects $\mathbb{C}$ at w.

$(\frac{2}{200})$

(a) You may assume that α is a right angle ie. $\alpha = \frac{\pi}{2}$. Show that $\gamma = \beta$.

Solution: Observe that the three angles of $\triangle N0z$ are $\pi/2$, θ, and β. Thus, $\beta = \pi/2 - \theta$. Observe that the three angles of $\triangle NS\zeta$ are $\alpha = \pi/2$, θ, and γ. Thus, $\gamma = \pi/2 - \theta$. Thus, $\gamma = \beta$.

$(\frac{2}{200})$

(b) Now show that $\delta = \gamma$.

Solution: $\triangle N0w$ is just the reflection of $\triangle S0w$ across the horizontal line. Angle δ is the image of γ under this reflection. Hence $\delta = \gamma$.

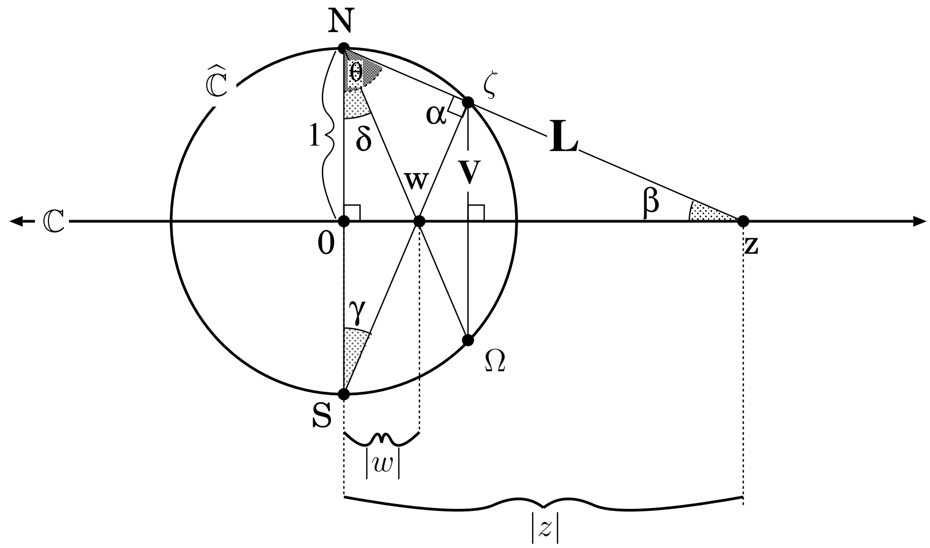

Figure 2: Cross section of Riemann sphere and complex plane.

$(\frac{2}{200})$ (c) Conclude that the triangle $\triangle N0w$ is *similar* to the triangle $\triangle z0N$.

Solution: The two of the three angles of $\triangle z0N$ are $\pi/2$ and β. Thus, $\theta = \frac{\pi}{2} - \beta$.

The two of the three angles of $\triangle N0w$ are $\frac{\pi}{2}$ and $\delta = \gamma$. The third must therefore equal $\frac{\pi}{2} - \beta = \theta$.

Thus, $\triangle z0N$ and $\triangle N0w$ have the same angles, so they must be similar triangles.

$(\frac{2}{200})$ (d) Conclude that $|w| \;=\; \dfrac{1}{|z|}$.

Solution: Because $\triangle N0w$ is similar to $\triangle z0N$, we have

$$\frac{|\overline{0w}|}{|\overline{N0}|} \quad = \quad \frac{|\overline{N0}|}{|\overline{0z}|}$$

But $|w| = |\overline{0w}|$ and $|z| = |\overline{0z}|$, while $|\overline{N0}| = 1$ (because $\widehat{\mathbb{C}}$ is a sphere of radius 1). Thus, we can rewrite this equation as: $|w| \;=\; \dfrac{1}{|z|}$.

$(\frac{12}{200})$ (e) Define $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = 1/\overline{z}$. Let $\widehat{f} : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ be the map which 'reflects' the Riemann sphere across the complex plane (eg. Figure 2, $\widehat{f}(\zeta) = \Omega$).

For any $z \in \mathbb{C}$, let $\widehat{z} \in \widehat{\mathbb{C}}$ be its image under the stereographic projection (eg. in Figure 2, $\widehat{z} =??$ and $\widehat{w} =??$).

Prove that $\widehat{f(z)} \;=\; \widehat{f}(\widehat{z})$.

Solution: In the picture, $\zeta = \widehat{z}$ and $\Omega = \widehat{w}$. In part 6d we established that $|w| \;=\; \dfrac{1}{|z|}$. Since $\arg(w) \;=\; \arg(z)$ (they are both on the same line through the origin), it follows that $w \;=\; 1/\overline{z} \;=\; f(z)$. Hence, we have: $\widehat{f(z)} \;=\; \widehat{w} \;=\; \Omega \;=\; \widehat{f}(\zeta) \;=\; \widehat{f}(\widehat{z})$.