Math 307 (*Complex Analysis*) — Final Exam 2004-12-21

- There are *seven* questions, worth a total of 200 marks. You have three hours. All questions are similar/identical to homework or practice problems. There are no 'trick questions'.
- *Justify your answers.* Show all steps in your computations. Please indicate your final answer by putting a $\boxed{\text{box}}$ around it (if applicable).
- You may invoke any theorem proved in class (eg. Cauchy-Riemann Equations, Cauchy's Theorem, Cauchy Integral Formula, Liouville's Theorem, Rouché's theorem, Maximum Modulus Principle, Open Mapping Theorem, etc.) Please clearly identify these results when you use them.
- Please write neatly and legibly. Illegible answers will not be graded charitably.
- Please *ask* about anything which confuses you.

1. Let $z_0 \in \mathbb{C}$ and let $\Gamma \subset \mathbb{C}$ be any counterclockwise contour around z_0.

$(\frac{5}{200})$ (a) Compute $\oint_\Gamma (z - z_0)^n \, dz$, for all $n \in \mathbb{N} := \{0, 1, 2, 3, \ldots\}$.

$(\frac{5}{200})$ (b) Compute $\oint_\Gamma \frac{1}{z - z_0} \, dz$.

$(\frac{4}{200})$ (c) Let $\mathbb{C}^* = \mathbb{C} \setminus \{z_0\}$. For any $n \in \mathbb{N}$, $n \geq 2$, define $f_n : \mathbb{C}^* \longrightarrow \mathbb{C}$ by $f_n(z) := \frac{1}{(z - z_0)^n}$. Find an analytic function $g : \mathbb{C}^* \longrightarrow \mathbb{C}$ such that $f_n(z) = g'(z)$ for all $z \in \mathbb{C}^*$.

$(\frac{6}{200})$ (d) Compute $\oint_\Gamma \frac{1}{(z - z_0)^n} \, dz$, for all $n \in \mathbb{N}$ with $n \geq 2$.

2. Let $z_0 \in \mathbb{C}$ and $0 < r < R$. Let $\mathbb{A} := \{z \in \mathbb{C} \,;\, r < |z - z_0| < R\}$ be an annulus around z_0. Let $f(z) := \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ be a Laurent series which converges on $\mathbb{A}$.

$(\frac{10}{200})$ (a) Let $\Gamma \subset \mathbb{A}$ be any counterclockwise contour around z_0. Compute $\oint_\Gamma f(z) \, dz$.

(**Hint:** *Use #1. You may interchange the "$\sum$" and the "$\int$" if necessary.*)

$(\frac{10}{200})$ (b) Suppose $f(z) = 0$ for all $z \in \mathbb{A}$. Show that $a_k = 0$ for all $k \in \mathbb{Z}$.

$(\frac{10}{200})$ (c) Let $g : \mathbb{A} \longrightarrow \mathbb{C}$ be an analytic function. Show that the Laurent series of g is *unique*. In other words, suppose that g has two Laurent series on $\mathbb{A}$:

$$g(z) := \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n, \ \forall\, z \in \mathbb{A}; \qquad \text{and} \qquad g(z) := \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \ \forall\, z \in \mathbb{A}.$$

for some coefficients $\{b_n\}_{n=-\infty}^{\infty}$ and $\{c_n\}_{n=-\infty}^{\infty}$. Show that $b_n = c_n$ for all $n \in \mathbb{Z}$.

$(\frac{15}{200})$ 3. Starting from the definitions of sin and cos [ie. $\frac{1}{2\mathbf{i}}(e^{\mathbf{i}\theta} \pm \text{etc.})$], show:

$$\text{For all } z \in \mathbb{C}, \quad \cos(z)^2 + \sin(z)^2 \quad = \quad 1.$$

4. Let $f(z) := \frac{z^4}{z^8 + 1}$.

$(\frac{10}{200})$ (a) Find all poles of f in $\mathbb{C}$.

$(\frac{10}{200})$ (b) Compute the residues of f at *some* (eg. half) of these poles (of your choice).

$(\frac{10}{200})$ (c) Compute $\int_{-\infty}^{\infty} \frac{x^4}{x^8 + 1} \, dx$. Carefully explain/justify any 'shortcuts' you use.

5. Let $\mathcal{U}$ be a simply connected domain bounded by a contour Γ. Suppose $\mathcal{D} \subset \mathbb{C}$ is a domain containing $\mathcal{U}$ and Γ, and let $f : \mathcal{U} \longrightarrow \mathbb{C}$ be analytic.

$(\frac{15}{200})$ (a) Suppose $f(z) \neq 0$ for all $z \in \mathcal{U}$. Show that $|f|$ has no local minima inside $\mathcal{U}$.

$(\frac{15}{200})$ (b) Suppose $|f|$ is constant on Γ. Show that *either* f has a zero inside $\mathcal{U}$, *or* f is constant.

6. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$, with $f(z) = u(z) + \mathbf{i}v(z)$ for some functions $u, v : \mathbb{C} \longrightarrow \mathbb{R}$.

$(\frac{15}{200})$ (a) Recall that $\nabla u := (\partial_x u,\ \partial_y u)$ and $\nabla v := (\partial_x v,\ \partial_y v)$.
Recall also that, if $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are two vectors in $\mathbb{R}^2$, then $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2$, and $\mathbf{u}$ and $\mathbf{v}$ are *orthogonal* if $\mathbf{u} \bullet \mathbf{v} = 0$.
Suppose f is analytic. Show that $\nabla u(z)$ is orthogonal to $\nabla v(z)$, for all $z \in \mathbb{C}$.

$(\frac{10}{200})$ (b) Let $z_0 \in \mathbb{C}$, and Let $f(z_0) = a + b\mathbf{i}$, and consider the level curves

$$\mathcal{A} \ := \ \{z \in \mathbb{C} \ ; \ u(z) = a\}; \quad \text{and} \quad \mathcal{B} \ := \ \{z \in \mathbb{C} \ ; \ v(z) = b\};$$

Suppose $f'(z_0) \neq 0$. Show that $\mathcal{A}$ and $\mathcal{B}$ intersect orthogonally at z_0.

7. [Recall: if $h : \mathbb{C} \longrightarrow \mathbb{C}$ is an analytic function, and $h(z_0) = 0$, then we write $\mathsf{order}\,(h; z_0) = n$ if $h(z) = (z - z_0)^n H(z)$, where H is analytic near z_0 and $H(z_0) \neq 0$. We say that $\mathsf{order}\,(h; \infty) = n$ if $\mathsf{order}\,(g; 0) = n$, where $g(z) = h(\frac{1}{z})$.]

Let $p, q : \mathbb{C} \longrightarrow \mathbb{C}$ be polynomials, and let $f(z) = \frac{p(z)}{q(z)}$ be a rational function, which we regard as a function $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ (where $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ is the Riemann sphere).

$(\frac{5}{200})$ (a) Suppose that $p(z) \ = \ p_c z^c + \cdots + p_1 z + p_0$, and $q(z) \ = \ q_d z^d + \cdots + q_1 z + q_0$, where $c := \deg(p)$ and $d := \deg(q)$. Let $g(z) = f(\frac{1}{z})$. Show that

$$g(z) \ = \ \frac{p_c z^d + p_{c-1} z^{d+1} + \cdots + p_1 z^{c+d-1} + p_0 z^{c+d}}{q_d z^c + q_{d-1} z^{c+1} + \cdots + q_1 z^{d+c-1} + q_0 z^{d+c}}.$$

$(\frac{5}{200})$ (b) Suppose $c < d$, and let $k = d - c$. Show that f has a zero of order k at ∞.

$(\frac{10}{200})$ (c) Show that, for any $z_0 \in \mathbb{C}$,

$$\Big(\ p(z_0) = 0\ \Big) \iff \Big(\ f(z_0) = 0\ \Big), \quad \text{and} \quad \mathsf{order}\,(p; z_0) = \mathsf{order}\,(f; z_0).$$

Conclude that $\#(\text{zeros of } f \text{ in } \mathbb{C}) = \#(\text{zeros of } p \text{ in } \mathbb{C})$ (counting multiplicity).
(**Hint:** *Assume that p and q have no zeros in common (otherwise we can just divide out a common factor).*)

$(\frac{5}{200})$ (d) Suppose $c \geq d$. Show that, in this case, for any $z_0 \in \widehat{\mathbb{C}}$ (including $z_0 = \infty$), we have

$$\Big(\ p(z_0) = 0\ \Big) \iff \Big(\ f(z_0) = 0\ \Big), \quad \text{and} \quad \mathsf{order}\,(p; z_0) = \mathsf{order}\,(f; z_0).$$

Conclude that $\#(\text{zeros of } f \text{ in } \widehat{\mathbb{C}}) = \#(\text{zeros of } p \text{ in } \widehat{\mathbb{C}})$ (counting multiplicity).

$(\frac{8}{200})$ (e) Define $\deg(f) := \max\{c, d\}$ (where $c := \deg(p)$ and $d := \deg(q)$).
Suppose $\deg(f) = D$. Show that $\#(\text{zeros of } f \text{ in } \widehat{\mathbb{C}}) = D$ (counting multiplicity).

$(\frac{7}{200})$ (f) Fix $\alpha \in \mathbb{C}$, and let $g(z) = f(z) - \alpha$. Show that $g(z)$ is also a rational function, and that $\deg(g) = \deg(f)$.

$(\frac{5}{200})$ (g) Suppose $D = \deg(f)$. For any $\alpha \in \mathbb{C}$, show that the equation "$f(z) = \alpha$" has exactly D solutions in $\widehat{\mathbb{C}}$ (counting multiplicity).
[In other words, f is exactly D-to-1 onto all $\alpha \neq \infty$. Indeed, we can use a similar argument to show that f is also exactly D-to-1 onto ∞. Hence $f : \widehat{\mathbb{C}} \longrightarrow \widehat{\mathbb{C}}$ is D-to-1 everywhere.]

$(\frac{5}{200})$ (h) Conclude: $\Big(\ f \text{ is injective on } \mathbb{C}\ \Big) \iff \Big(\ f \text{ is a linear fractional transformation}\ \Big)$.
[In other words, *Linear fractional transformations are the only rational functions which are conformal isomorphisms of the Riemann sphere.*]