Math 426H (*Differential Geometry*) — Final Exam —April 24, 2006.

1. Let $\mathcal{M}$ be a surface and let $\beta : [0,1] \longrightarrow \mathcal{M}$ be a smooth loop. Let ϕ be a 1-form on $\mathcal{M}$.

$(\frac{8}{200})$ (a) Suppose ϕ is *exact* (i.e. $\phi = df$ for some $f : \mathcal{M} \longrightarrow \mathbb{R}$). Show that $\int_\beta \phi = 0$.

Solution: $\int_\beta \phi \;=\; \int_\beta df \underset{(*)}{=} f[\beta(1)] - f[\beta(0)] \underset{(\dagger)}{=} f[f(0)] - f[f(0)] \;=\; 0.$

Here, $(*)$ is the 'Fundamental Theorem of Calculus' for 1-forms (Theorem 4.6.2, p.170), while $(\dagger)$ is because β is a loop, so $\beta(0) = \beta(1)$. □

$(\frac{8}{200})$ (b) Let $\mathcal{M} := \mathbb{R}^2 \setminus \{(0,0)\}$ be the 'punctured plane'. Compute $d\phi$, where

$$\phi \;:=\; f(x,y)\,dy - g(x,y)\,dx, \quad \text{with} \quad f(x,y) \;:=\; \frac{x}{x^2+y^2} \quad \text{and} \quad g(x,y) \;:=\; \frac{y}{x^2+y^2}.$$

Solution: First, observe that $\partial_x f(x,y) \;=\; \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} \;=\; \frac{x^2-y^2}{(x^2+y^2)^2}.$

By switching x with y, we also get $\partial_y f(x,y) \;=\; \frac{y^2-x^2}{(x^2+y^2)^2}$. Thus,

$$d\phi \;=\; \partial_x f(x,y)\,dxdy - \partial_y g(x,y)\,dydx \;=\; (\partial_x f(x,y) + \partial_y g(x,y))\,dxdy$$
$$=\; \frac{(x^2-y^2)+(y^2-x^2)}{(x^2+y^2)^2}\,dxdy \;=\; \boxed{0.}$$

□

$(\frac{8}{200})$ (c) Let $\beta : [0, 2\pi] \longrightarrow \mathcal{M}$ be the *unit circle*: $\beta(t) := (\cos(t), \sin(t))$. Compute $\int_\beta \phi$.

Solution: If $t \in [0, 2\pi]$, then $\beta'(t) = (-\sin(t), \cos(t))$ and $\phi(\beta(t)) = \cos(t)\,dy - \sin(t)\,dx$. Thus,

$$\phi[\beta'(t)] \;=\; [\cos(t)\,dy - \sin(t)\,dx](-\sin(t), \cos(t)) \;=\; \sin(t)^2 + \cos(t)^2 \;=\; 1.$$

Thus, $\int_\beta \phi \;=\; \int_0^{2\pi} \phi[\beta'(t)]\,dt \;=\; \int_0^{2\pi} 1\,dt \;=\; \boxed{2\pi.}$ □

$(\frac{10}{200})$ (d) Is ϕ closed? Is ϕ exact? Is β nullhomotopic in $\mathcal{M}$? *Justify* your answers.

Solution: ϕ is $\boxed{\text{closed}}$ because part (b) says $d\phi = 0$.

ϕ is $\boxed{\text{not exact}}$ because if it was, then (a) would imply that $\int_\gamma \phi = 0$, which contradicts (c).

Likewise, γ is $\boxed{\text{not nullhomotopic}}$, because if it was, then Lemma 4.7.8 (p.182) would imply that $\int_\gamma \phi = 0$, which contradicts (c).

(The existence of a *nonexact* closed 1-form shows that the *de Rham cohomology group* $\mathcal{H}^1(\mathcal{M})$ is nontrivial. The existence of a loop which is *not* nullhomotopic shows that the *fundamental group* $\pi_1(\mathcal{M})$ is nontrivial. Each of these indicates the presence of the 'hole' in $\mathcal{M}$.) □

$(\frac{6}{200})$ (e) Let $\mathbf{L} := \{(x,0)\,;\, x \geq 0\}$ be the positive 'x' axis, and let $\mathcal{N} := \mathbb{R}^2 \setminus \mathbf{L}$ be the plane with this line removed, as in Figure 1(A) below [thus, $\mathcal{N} \subset \mathcal{M}$]. Is ϕ exact when restricted to $\mathcal{N}$? Why or why not? (You may use any theorem from the text).

Solution: $\boxed{\text{Yes,}}$ ϕ is exact on $\mathcal{N}$. The surface $\mathcal{N}$ is *simply connected*. Thus, any closed 1-form on $\mathcal{N}$ is exact, by Poincaré's Lemma (Lemma 4.7.9, p.183). Thus, ϕ is exact on $\mathcal{N}$, because ϕ is closed by part (b). □

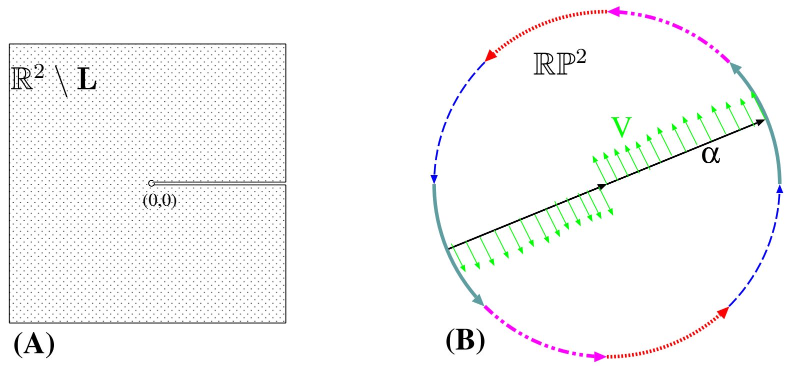

Figure 1: **(A)** Question #1(e) **(B)** Question #4(c)

2. Let $\mathcal{M}$ be a surface and let $\mathbf{p} \in \mathcal{M}$. Suppose $\vec{\mathbf{v}}_\mathbf{p}, \vec{\mathbf{w}}_\mathbf{p} \in \mathsf{T}_\mathbf{p}\mathcal{M}$ are two *orthonormal* tangent vectors, and that $S(\vec{\mathbf{v}}_\mathbf{p}) \bullet S(\vec{\mathbf{w}}_\mathbf{p}) = 0$. What can you conclude about any special geometric properties of $\vec{\mathbf{v}}_\mathbf{p}$ and $\vec{\mathbf{w}}_\mathbf{p}$, or about the principal/Gaussian/mean curvatures of $\mathcal{M}$ at $\mathbf{p}$?

$(\frac{30}{200})$

Solution: We conclude that $\boxed{\text{either } H = 0, \text{ or } \vec{\mathbf{v}}_\mathbf{p} \text{ and } \vec{\mathbf{w}}_\mathbf{p} \text{ are principal vectors}}$.

Case 1: If $\mathbf{p}$ is an umbilic point, then every vector is a principal vector (including $\vec{\mathbf{v}}_\mathbf{p}$ and $\vec{\mathbf{w}}_\mathbf{p}$). In this case, S is just multiplication by some scalar k, so $S(\vec{\mathbf{v}}_\mathbf{p}) = k\vec{\mathbf{v}}_\mathbf{p}$ and $S(\vec{\mathbf{w}}_\mathbf{p}) = k\vec{\mathbf{w}}_\mathbf{p}$, so that

$$S(\vec{\mathbf{v}}_\mathbf{p}) \bullet S(\vec{\mathbf{w}}_\mathbf{p}) \quad = \quad (k\vec{\mathbf{v}}_\mathbf{p}) \bullet (k\vec{\mathbf{w}}_\mathbf{p}) \quad = \quad k^2 \vec{\mathbf{v}}_\mathbf{p} \bullet \mathbf{w} \quad = \quad k^2 0 \quad = \quad 0,$$

Case 2: Suppose that $\mathbf{p}$ is *not* an umbilic point. Let $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ be the (unique) principal vectors at $\mathbf{p}$; then $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ are orthonormal eigenvectors of S, so the matrix of S relative to the basis $\{\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2\}$ is $\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$ for some $k_1 > k_2$ (note that $k_1 \neq k_2$ because we assume $\mathbf{p}$ not umbilic). Let $\vec{\mathbf{v}}_\mathbf{p} = v_1\vec{\mathbf{e}}_1 + v_2\vec{\mathbf{e}}_2$ and $\vec{\mathbf{w}}_\mathbf{p} = w_1\vec{\mathbf{e}}_1 + w_2\vec{\mathbf{e}}_2$ for some $v_1, v_2, w_1, w_2 \in \mathbb{R}$. Then

$$S(\vec{\mathbf{v}}_\mathbf{p}) \quad = \quad S(v_1\vec{\mathbf{e}}_1 + v_2\vec{\mathbf{e}}_2) \quad = \quad v_1 S(\vec{\mathbf{e}}_1) + v_2 S(\vec{\mathbf{e}}_2) \quad = \quad k_1 v_1 \vec{\mathbf{e}}_1 + k_2 v_2 \vec{\mathbf{e}}_2.$$

Likewise, $S(\vec{\mathbf{w}}_\mathbf{p}) = k_1 w_1 \vec{\mathbf{e}}_1 + k_2 w_2 \vec{\mathbf{e}}_2$. Thus,

$$\begin{aligned} S(\vec{\mathbf{v}}_\mathbf{p}) \bullet S(\vec{\mathbf{w}}_\mathbf{p}) &= (k_1 v_1 \vec{\mathbf{e}}_1 + k_2 v_2 \vec{\mathbf{e}}_2) \bullet (k_1 w_1 \vec{\mathbf{e}}_1 + k_2 w_2 \vec{\mathbf{e}}_2) \\ &= k_1^2 v_1 w_1 \vec{\mathbf{e}}_1 \bullet \vec{\mathbf{e}}_1 + k_1 k_2 v_1 w_2 \vec{\mathbf{e}}_1 \bullet \vec{\mathbf{e}}_2 + k_1 k_2 v_2 w_1 \vec{\mathbf{e}}_2 \bullet \vec{\mathbf{e}}_1 + k_2^2 v_2 w_2 \vec{\mathbf{e}}_2 \bullet \vec{\mathbf{e}}_2 \\ &\underset{(\dagger)}{=} k_1^2 v_1 w_1 + k_2^2 v_2 w_2 \quad = \quad k_1^2 v_1 w_1 + k_1^2 v_2 w_2 + (k_1^2 - k_2^2) v_2 w_2 \\ &= k_1^2 (v_1 w_1 + v_2 w_2) + (k_1^2 - k_2^2) v_2 w_2 \\ &= k_1^2 (\vec{\mathbf{v}}_\mathbf{p} \bullet \vec{\mathbf{w}}_\mathbf{p}) + (k_1^2 - k_2^2) v_2 w_2 \quad \underset{(*)}{=} \quad (k_1^2 - k_2^2) v_2 w_2. \end{aligned}$$

where $(\dagger)$ is because $\vec{\mathbf{e}}_i \bullet \vec{\mathbf{e}}_j = \delta_{ij}$, and $(*)$ is because $\vec{\mathbf{v}}_\mathbf{p} \bullet \vec{\mathbf{w}}_\mathbf{p} = 0$ because $\vec{\mathbf{v}}_\mathbf{p}$ is orthogonal to $\vec{\mathbf{w}}_\mathbf{p}$. Thus,

$$v_2 w_2 \quad = \quad \frac{S(\vec{\mathbf{v}}_\mathbf{p}) \bullet S(\vec{\mathbf{w}}_\mathbf{p})}{k_1^2 - k_2^2} \quad = \quad \frac{0}{k_1^2 - k_2^2} \quad = \quad 0.$$

At this point there are two possibilities:

Case 2(a): Suppose $k_1^2 - k_2^2 = 0$. Then $(k_1 + k_2)(k_1 - k_2) = 0$. Thus, $H = k_1 + k_2 = 0$, because $k_1 - k_2$ cannot be zero, because $k_1 \neq k_2$ (because we assume $\mathbf{p}$ is not umbilic).

Case 2(b): If $k_1^2 - k_2^2 \neq 0$, then either $v_2 = 0$ or $w_2 = 0$. Suppose that $v_2 = 0$; then $v_1 = 1$ (because $v_1^2 + v_2^2 = \|\vec{\mathbf{v}}_{\mathbf{p}}\|^2 = 1$), so that $\vec{\mathbf{v}}_{\mathbf{p}} = \vec{\mathbf{e}}_1$. But then $\vec{\mathbf{w}}_{\mathbf{p}} = \pm\vec{\mathbf{e}}_2$ because $\vec{\mathbf{w}}_{\mathbf{p}}$ is normal to $\vec{\mathbf{v}}_{\mathbf{p}} = \vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ is also normal to $\vec{\mathbf{e}}_1$.

We conclude that $\vec{\mathbf{v}}_{\mathbf{p}} = \vec{\mathbf{e}}_1$ and $\vec{\mathbf{w}}_{\mathbf{p}} = \vec{\mathbf{e}}_2$ are the two principal vectors. □

$\left(\frac{25}{200}\right)$ 3. Let $\mathcal{M}, \overline{\mathcal{M}} \subset \mathbb{R}^3$ be two surfaces, and let $f : \mathcal{M} \longrightarrow \overline{\mathcal{M}}$ be a local isometry. Let $\alpha : [0,1] \longrightarrow \mathcal{M}$ be a unit-speed curve, and let $\overline{\alpha} := f \circ \alpha : [0,1] \longrightarrow \overline{\mathcal{M}}$. Show that

$$\Big(\alpha \text{ is a geodesic on } \mathcal{M}\Big) \iff \Big(\overline{\alpha} \text{ is a geodesic on } \overline{\mathcal{M}}\Big).$$

(You may use any result from previous quizzes.)

Solution: Let $(\vec{E}_1, \vec{E}_2)$ be a tangent frame field on α so that $\vec{E}_1 = \alpha'$ and $\vec{E}_2$ is orthogonal to $\vec{E}_1$. Let $(\underline{E}_1, \underline{E}_2)$ be the corresponding frame field on $\overline{\alpha}$ —i.e. for all $t \in [0,1]$, $\underline{E}_1 = F_*(\vec{E}_1) = \overline{\alpha}'(t)$, and $\underline{E}_2 = F_*(\vec{E}_2)$; hence $(\underline{E}_1, \underline{E}_2)$ is a frame field along $\overline{\alpha}$, because F_* preserves inner products.

Let ω_{12} be the connection form for $(\vec{E}_1, \vec{E}_2)$ and let $\overline{\omega}_{12}$ be the connection form for $(\underline{E}_1, \underline{E}_2)$. Then Lemma 6.5.3(b) says that $\omega_{12} = F^*\overline{\omega}_{12}$. Thus,

$$\Big(\alpha \text{ is a geodesic on } \mathcal{M}\Big) \underset{(*)}{\iff} \Big(\omega_{12}(\alpha') \equiv 0\Big) \iff \Big(F^*\overline{\omega}_{12}(\alpha') \equiv 0\Big) \iff \Big(\overline{\omega}_{12}(F_*\alpha') \equiv 0\Big)$$
$$\iff \Big(\overline{\omega}_{12}(\overline{\alpha}') \equiv 0\Big) \underset{(*)}{\iff} \Big(\overline{\alpha} \text{ is a geodesic on } \overline{\mathcal{M}}\Big).$$

Here, $(*)$ is by exercise #6.1.1 (which appeared on Quiz #9).

① Some people tried to map the second derivative through the tangent map. In other words, they wrote expressions like "$\overline{\alpha}'' = F_*(\alpha'')$." There are two problems with this:

- α'' is not necessarily an element of $\mathsf{T}_{\mathbf{p}}\mathcal{M}$ (indeed, if α is a geodesic, then α'' is *normal* to $\mathsf{T}_{\mathbf{p}}\mathcal{M}$). Thus, $F_*(\alpha'')$ is not well-defined, because F_* is only defined for tangent vectors.
- The (correct) equation "$\overline{\alpha}' = F_*(\alpha')$" is just a special case of the Chain Rule. It can be true because α' and F_* are both *first* derivatives. However, α'' is a *second* derivative. Hence, if α'' *was* a tangent vector, then the proper equation would be something like "$\overline{\alpha}'' = F_{**}(\alpha'')$", where F_{**} is the *second* derivative of F —that is, the derivative of F_* —that is, the tangent map of the *tangent map* (i.e. the 'double tangent map').

In fact, F_{**} is a well-defined object, but it is more complicated than you think. Let's adopt the old notation $F_*(\mathbf{p}) := \mathsf{T}_{\mathbf{p}}F$. Recall that the *tangent bundle* $\mathsf{T}\mathcal{M}$ is a four-dimensional smooth manifold. If we 'glue together' the tangent maps $\mathsf{T}_{\mathbf{p}}F : \mathsf{T}_{\mathbf{p}}\mathcal{M} \longrightarrow \mathsf{T}_{F(\mathbf{p})}\overline{\mathcal{M}}$ for all $\mathbf{p} \in \mathcal{M}$, then we get a function $\mathsf{T}F : \mathsf{T}\mathcal{M} \longrightarrow \mathsf{T}\overline{\mathcal{M}}$, and $\mathsf{T}F$ is a smooth mapping between these two four-dimensional manifolds. Thus, we can then compute the tangent map of $\mathsf{T}F$ itself, to get a function $\mathsf{T}\mathsf{T}F : \mathsf{T}\mathsf{T}\mathcal{M} \longrightarrow \mathsf{T}\mathsf{T}\overline{\mathcal{M}}$. This is a smooth function between the *double*-tangent bundles (which are smooth, *eight*-dimensional manifolds) and it is the correct generalization of the 'second derivative' for smooth mappings on manifolds.

② Many people argued as follows: "α is a geodesic iff α'' is orthogonal to $\mathcal{M}$. If F is an isometry, then F preserves inner products; hence $F_*\alpha''$ will be orthogonal to $\overline{\mathcal{M}}$."

The problem here is that F_* only preserves inner products *between tangent vectors*. Indeed, F_* is only *defined* on tangent vectors. If α'' is orthogonal to $\mathcal{M}$, then $F_*\alpha''$ isn't even well-defined.

In princial, we *could* well-defined $F_*\alpha''$ by smoothly extending F from $\mathcal{M}$ to a smooth map $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$. In this case, $F_*\alpha''$ would be well-defined (because α'' *is* a tangent vector to $\mathbb{R}^3$).

However, the fact that F is an isometry only guarantees that F_* preserves inner products (and hence, orthogonality) between vectors tangent to the *surface* $\mathcal{M}$, and α'' is *not* tangent to $\mathcal{M}$, so there is no guarantee that $F_*(\alpha'')$ will be orthogonal to anything.

③ Some people wrote down basically the correct proof, but they forgot to define the frame fields $(\vec{E}_1, \vec{E}_2)$ and $(\underline{E}_1, \underline{E}_2)$. In other words, they just said, "Let ω_{12} be the connection 1-form of $\mathcal{M}$..." and went from there. The problem is that "the connection 1-form" is only well-defined *with respect to a specific frame field.* There is not such thing as "the connection 1-form of $\mathcal{M}$" —you can only talk about "the connection 1-form of the frame field $(\vec{E}_1, \vec{E}_2)$ defined on $\mathcal{M}$". □

4. Let $\mathcal{M}$ be an abstract surface.

$(\frac{5}{200})$ (a) Define what it means for $\mathcal{M}$ to be *orientable*.

Solution: $\mathcal{M}$ is orientable if there is a nonvanishing smooth 2-form η on $\mathcal{M}$. Here, *nonvanishing* means that, for all $\mathbf{p} \in \mathcal{M}$, and any pair of linearly independent tangent vectors $\vec{\mathbf{v}}_{\mathbf{p}}, \vec{\mathbf{w}}_{\mathbf{p}} \in \mathsf{T}_{\mathbf{p}}\mathcal{M}$, $\eta[\vec{\mathbf{v}}_{\mathbf{p}}, \vec{\mathbf{w}}_{\mathbf{p}}] \neq 0$.

④ Some people said, "$\mathcal{M}$ is orientable if there is a nonvanishing normal vector field on $\mathcal{M}$". This definition is correct, but *only for surfaces embeddeded in* $\mathbb{R}^3$. The problem is that 'normal vector fields' are only well-defined on surfaces embedded in $\mathbb{R}^3$, and here, $\mathcal{M}$ is an *abstract* surface. This is an important distinction, because some abstract surfaces [such as the real projective plane $\mathbb{RP}^2$ in part (c)] simply *cannot* be embedded in $\mathbb{R}^3$. Thus, it makes no sense to talk about putting a normal vector field on $\mathbb{RP}^2$ (nonvanishing or otherwise). In other words, you just can't do part (c) unless you use the 'abstract surface' definition of orientability. □

$(\frac{15}{200})$ (b) Suppose there is a smooth closed curve $\alpha : [0,1] \longrightarrow \mathcal{M}$ [i.e. $\alpha(0) = \alpha(1)$ and $\alpha'(0) = \alpha'(1)$] and a smooth tangent vector field $\vec{V} : [0,1] \longrightarrow \mathsf{T}\mathcal{M}$ defined along γ, such that:

i. $\vec{V}(t)$ and $\alpha'(t)$ are linearly independent for all $t \in [0,1]$.

ii. $\vec{V}(1) = -\vec{V}(0)$.

Show that $\mathcal{M}$ is *not* orientable.

Solution: Suppose $\mathcal{M}$ was orientable, and let η be a nonvanishing two-form. Define the function $f : [0,1] \longrightarrow \mathbb{R}$ by $f(t) := \eta[\alpha'(t), \vec{V}(t)]$. Then f is a smooth function (because α', $\vec{V}$, and η are smooth) and $f(t) \neq 0$ for all t by hypothesis (i), because η is nonvanishing. However, hypothesis (ii) implies that

$$f(1) \quad = \quad \eta[\alpha'(1), \vec{V}(1)] \quad = \quad \eta[\alpha'(0), -\vec{V}(0)] \quad = \quad -\eta[\alpha'(0), \vec{V}(0)] \quad = \quad -f(0).$$

Thus, if $f(0) > 0$, then $f(1) < 0$, whereas if $f(0) < 0$, then $f(1) > 0$. Either way, the Intermediate Value Theorem says that $f(t) = 0$ for some $t \in (0,1)$. Contradiction. □

$(\frac{10}{200})$ (c) Figure 1(B) is a picture of the *real projective plane* $\mathbb{RP}^2$ as a disk with opposite boundary points identified. Draw a smooth closed path on $\mathbb{RP}^2$ and a smooth tangent vector field along this path, satisfying the hypothesis of part (b). Conclude that $\mathbb{RP}^2$ is *not* orientable.

Solution: See Figure 1(B).

⑤ Some people drew some pretty weird pictures, where it really wasn't clear which way the vector field was pointing, etc. Indeed, several people drew a path along the 'perimeter' of the disk in the picture. This is a bad idea, because every point in $\mathbb{RP}^2$ corresponds to *two* points on the perimeter. $\square$

5. Let $\vec{V}$ and $\vec{W}$ be two smooth tangent vector fields on $\mathcal{M}$. Let $\vec{U}$ be a unit normal vector field, and let $S : \mathsf{T}\mathcal{M} \longrightarrow \mathsf{T}\mathcal{M}$ be the shape operator.

$(\frac{10}{200})$ (a) Show that $\vec{U} \bullet \nabla_{\vec{V}} \vec{W} \;=\; S(\vec{V}) \bullet \vec{W}$.

Solution: First, note that $\vec{U} \bullet \vec{W} \equiv 0$ because $\vec{W}$ is a tangent field and $\vec{U}$ is a normal field. Thus,

$$0 \quad = \quad \vec{V}[0] \quad = \quad \vec{V}[\vec{U} \bullet \vec{W}] \underset{(L)}{=\!=} (\nabla_{\vec{V}} \vec{U}) \bullet \vec{W} \;+\; \vec{U} \bullet \nabla_{\vec{V}} \vec{W}.$$

Here (L) is the Liebniz rule for directional derivatives (Theorem 2.5.3(4) on p.79). Thus, $\vec{U} \bullet \nabla_{\vec{V}} \vec{W} \;=\; -(\nabla_{\vec{V}} \vec{U}) \bullet \vec{W} \;=\; S(\vec{V}) \bullet \vec{W}$. $\square$

$(\frac{15}{200})$ (b) The *Lie Bracket* of $\vec{V}$ and $\vec{W}$ is the vector field defined by:

$$[\vec{V}, \vec{W}] \quad := \quad \nabla_{\vec{V}} \vec{W} - \nabla_{\vec{W}} \vec{V}.$$

Show that $[\vec{V}, \vec{W}]$ is *also* a tangent vector field —i.e. show that $[\vec{V}, \vec{W}](\mathbf{p}) \in \mathsf{T}_{\mathbf{p}}\mathcal{M}$ for all $\mathbf{p} \in \mathcal{M}$. (**Hint:** *Use the symmetry of* S.)

Solution: $[\vec{V}, \vec{W}]$ is a tangent vector field iff $\vec{U}$ is orthogonal to $[\vec{V}, \vec{W}]$. But

$$\begin{aligned}\vec{U} \bullet [\vec{V}, \vec{W}] \quad &= \quad \vec{U} \bullet (\nabla_{\vec{V}} \vec{W} - \nabla_{\vec{W}} \vec{V}) \quad = \quad \vec{U} \bullet \nabla_{\vec{V}} \vec{W} - \vec{U} \bullet \nabla_{\vec{W}} \vec{V} \\ &\underset{(@)}{=\!=} \; S(\vec{V}) \bullet \vec{W} - S(\vec{W}) \bullet \vec{V} \underset{(\S)}{=\!=} S(\vec{V}) \bullet \vec{W} - S(\vec{V}) \bullet \vec{W} \quad = \quad 0.\end{aligned}$$

Here, (@) is by part (a) and (§) is because S is a symmetric operator, so $S(\vec{W}) \bullet \vec{V} = S(\vec{V}) \bullet \vec{W}$.

Thus, $\vec{U} \bullet [\vec{V}, \vec{W}] = 0$, so $[\vec{V}, \vec{W}]$ is a tangent vector field, as desired. $\square$

$(\frac{10}{200})$ 6. Let $\mathcal{M}$ be a torus, and let α be the 'top circle' of $\mathcal{M}$, as shown in Figure 2(A) above (ignore the scissors for now). Is α an *asymptotic curve* on $\mathcal{M}$? Why or why not?

Solution: Yes, α is an asymptotic curve, because α is a circle in a plane $\mathbb{P}$ tangent to $\mathcal{M}$, so the acceleration vector α'' is always in $\mathbb{P}$, and hence, always tangent to $\mathcal{M}$. Thus, α is asymptotic by the remarks on page 235. $\square$

7. Let $\mathcal{M} \subset \mathbb{R}^3$ be a surface and let $\gamma : [0,1] \longrightarrow \mathcal{M}$ be a unit-speed curve. Let $(\vec{T}, \vec{N}, \vec{B})$ be the *Frenet frame field* along γ; that is, $\vec{T} := \gamma'$ is the *tangent vector field*, $\vec{N}$ is the *normal vector field*, and $\vec{B}$ is the *binormal vector field*. Let τ be the *torsion* of the curve γ (as defined by the *Frenet equations*).

$(\frac{5}{200})$ (a) Show: $\Big(\gamma \text{ is an asymptotic curve}\Big) \iff \Big(\vec{B} \text{ is orthogonal to } \mathcal{M} \text{ at all points on } \gamma\Big)$.

Solution: $\Big(\gamma \text{ is an asymptotic curve}\Big) \underset{(*)}{\iff} \Big(\gamma'' \text{ is tangent to } \mathcal{M}\Big) \iff \Big(\vec{N} \text{ is tangent to } \mathcal{M}\Big)$ $\underset{(\dagger)}{\iff} \Big(\vec{B} \text{ is orthogonal to } \mathcal{M}\Big)$. Here, $(*)$ is by the criterion on p. 235.

$(\dagger)$ is because $\vec{T} \perp \vec{B} \perp \vec{N}$;' hence $\vec{T}$ and $\vec{N}$ are tangent to $\mathcal{M}$ iff $\vec{B}$ is normal to $\mathcal{M}$. $\square$

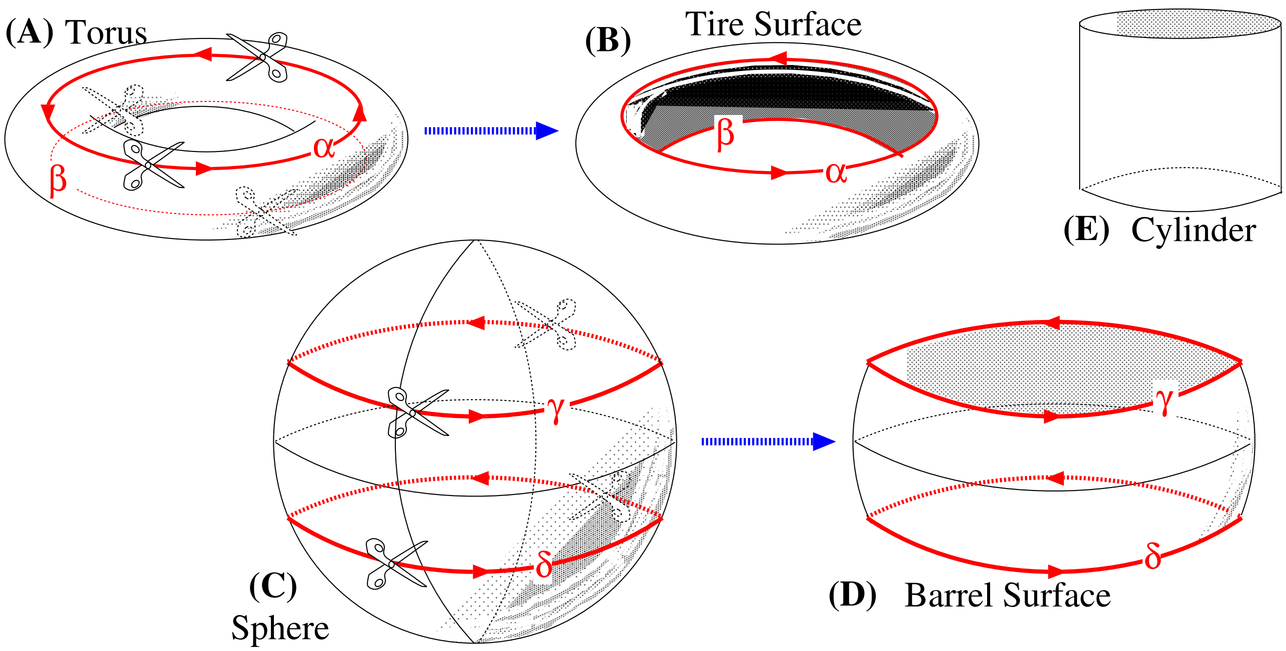

Figure 2: Questions #6, #7, and #8.

$(\frac{5}{200})$ (b) Suppose γ is asymptotic. Deduce from (a) that $S(\vec{T}) = \tau\vec{N}$ at all points on γ.

Solution: Part (a) says that $\vec{B}$ is a unit normal vector field along γ. Thus,

$$S(\vec{T}) \underset{(*)}{=} -\nabla_{\vec{T}}\vec{B} = -\vec{B}' \underset{(\dagger)}{=} \tau\vec{N}.$$

Here, $(*)$ is the definition of the shape operator (Defn. 5.1.1, p.196).

$(\dagger)$ is because $\vec{B}' = -\tau\vec{N}$ by the third Frenet formula (Theorem 2.3.2, p.58). □

$(\frac{10}{200})$ (c) Suppose γ is asymptotic. Show that $K = -\tau^2$ at all points on γ (where K is the Gaussian curvature of $\mathcal{M}$).

Solution: Fix $t \in [0,1]$. Let $\mathbf{p} := \gamma(t)$, and let $\vec{\mathbf{e}}_1$ and $\vec{\mathbf{e}}_2$ be the two principal vectors at $\mathbf{p}$. Thus, $S(\vec{\mathbf{e}}_1) = k_1\vec{\mathbf{e}}_1$ and $S(\vec{\mathbf{e}}_2) = k_2\vec{\mathbf{e}}_2$, where k_1, k_2 are the principal curvatures.

Suppose $\vec{T} = c\vec{\mathbf{e}}_1 + s\vec{\mathbf{e}}_2$, for some $c, s \in \mathbb{R}$; then $c^2 + s^2 = 1$, because $\vec{T}$ is a unit vector. Also $\vec{N} = \mp s\vec{\mathbf{e}}_1 \pm c\vec{\mathbf{e}}_2$, because $\vec{N}$ is a unit vector orthogonal to $\vec{T}$, and the only unit vectors orthogonal to (c, s) are $(-s, c)$ and $(s, -c)$. Assume WOLOG that $\vec{N} = -s\vec{\mathbf{e}}_1 + c\vec{\mathbf{e}}_2$; then

$$\begin{aligned} S(\vec{T}) &= S(c\vec{\mathbf{e}}_1 + s\vec{\mathbf{e}}_2) = cS(\vec{\mathbf{e}}_1) + sS(\vec{\mathbf{e}}_2) = ck_1\vec{\mathbf{e}}_1 + sk_2\vec{\mathbf{e}}_2, \\ \text{and also, } S(\vec{T}) &\underset{(*)}{=} \tau\vec{N} = -\tau s\vec{\mathbf{e}}_1 + \tau c\vec{\mathbf{e}}_2. \end{aligned}$$

Here $(*)$ is by part (b). This gives two different orthonormal expansions for $S(\vec{T})$ in terms of the basis $(\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2)$. But any vector has a *unique* orthonormal expansion, so we conclude that $ck_1 = -\tau s$ and $sk_2 = \tau c$. Hence

$$k_1 = \frac{-\tau s}{c} \quad \text{and} \quad k_2 = \frac{\tau c}{s}.$$

Hence $K = k_1 k_2 = \frac{-\tau s}{c} \cdot \frac{\tau c}{s} = -\tau^2$, as desired. □

$(\frac{10}{200})$ (d) Let $\mathcal{M}$ be a torus, and let α be the 'top circle' of $\mathcal{M}$ as in Figure 2(A) above (ignore the scissors). Compute the Gaussian curvature of $\mathcal{M}$ at points on α.

Solution: $\boxed{K \equiv 0 \text{ on } \alpha.}$ To see this, recall that α is an asymptotic curve by question #6. Thus, part (c) says that $K \equiv -\tau^2$ along α. However, the torsion of α is everywhere equal to zero, because α is a circle, so it lies entirely in a single plane; hence $\tau \equiv 0$ by Corollary 2.3.5 (p.61). □

$(\frac{10}{200})$ 8. Let α and β be the 'top' and 'bottom' circles of a torus, as shown in Figure 2(A). If we cut the torus along α and β, and remove the inner half, then we are left with the ***tire surface*** of Figure 2(B). Similarly, let γ and δ be two parallels of a unit sphere, as shown in Figure 2(C); if we cut the sphere along γ and δ, and remove the top and bottom 'domes', then we are left with the ***barrel surface*** of Figure 2(D). Finally, consider the ***cylinder*** in Figure 2(E).

Are there any isometries between any two surfaces in the collection {Tire, Barrel, Cylinder}? If so, then *describe* these isometries. If not, then *prove* that no such isometries can exist.

Solution: There are $\boxed{\text{no isometries}}$ from any member of {Tire, Barrel, Cylinder} to any other member. To see this, we use Theorema Egregium. First, recall that the cylinder has constant Gaussian curvature $K \equiv 0$. Also, the unit sphere has constant Gaussian curvature $K \equiv 1$, so the Barrel surface (a subset of the sphere) also has constant Gaussian curvature $K \equiv 1$. Thus, Theorema Egregium says the Barrel cannot be isometric to the Cylinder.

In #6(d) we showed that the Gaussian curvature of the torus along the curve α is equal to zero. Hence the Gaussian curvature of the Tire is arbitrarily close to zero near the boundaries α and β, so the Tire cannot be isometric to the Barrel (which has constant curvature $K \equiv 1$).

Note, however, that the Gaussian curvature of the Tire is never *equal* to zero. The curves α and β themselves are *not* part of the Tire. Indeed, the torus has positive Gaussian curvature on its outer side; hence the Tire has positive curvature everywhere; hence Theorema Egregium says the Tire is not isometric to the Cylinder. □