Math 310 (*Metric Spaces*) — Final Exam—April ??, 2006.

1. $(\frac{25}{100})$ Let $\mathcal{X}$ and $\mathcal{Y}$ be path-connected topological spaces. Let $x_0 \in \mathcal{X}$ and $y_0 \in \mathcal{Y}$. Let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ and let $z_0 = (x_0, y_0) \in \mathcal{Z}$. Show that $\pi_1(\mathcal{Z}, z_0) \cong \pi_1(\mathcal{X}, x_0) \times \pi_1(\mathcal{Y}, y_0)$.

   (**Hint:** *Define a function* $h : \pi_1(\mathcal{Z}, z_0) \longrightarrow \pi_1(\mathcal{X}, x_0) \times \pi_1(\mathcal{Y}, y_0)$. *Show that* h *is injective, surjective, and a group homomorphism.*)

Solution: Let $f : \mathcal{Z} \longrightarrow \mathcal{X}$ and $g : \mathcal{Z} \longrightarrow \mathcal{Y}$ be the coordinate projection maps [i.e. $f(x, y) = x$ and $g(x, y) = y$]. Note that $f(z) = x$ and $g(z) = y$. Let $f_* : \pi_1(\mathcal{Z}, z) \longrightarrow \pi_1(\mathcal{X}, x)$ and $g_* : \pi_1(\mathcal{Z}, z) \longrightarrow \pi_1(\mathcal{Y}, y)$ be the induced homomorphisms. Define $h : \pi_1(\mathcal{Z}, z_0) \longrightarrow \pi_1(\mathcal{X}, x_0) \times \pi_1(\mathcal{Y}, y_0)$ by $h(\alpha) = \Big(f_*(\alpha), g_*(\alpha)\Big)$. We will show that h is a group isomorphism.

Homomorphism: $h(\alpha\beta) = \Big(f_*(\alpha\beta), g_*(\alpha\beta)\Big) \underset{(*)}{=} \Big(f_*(\alpha)f_*(\beta), g_*(\alpha)f_*(\beta)\Big) \underset{(\dagger)}{=} \Big(f_*(\alpha), g_*(\alpha)\Big) \cdot (f_*(\beta), g_*(\beta)) = h(\alpha) \cdot h(\beta)$.

Here $(*)$ is because f_* and g_* are homomorphisms, and $(\dagger)$ is by definition of multiplication in the direct product of two groups.

Injective: Let $[\alpha], [\beta] \in \pi_1(\mathcal{Z}, z_0)$, and suppose $h[\alpha] = h[\beta]$; we must show that $[\alpha] = [\beta]$

If $h[\alpha] = h[\beta]$, then $f_*[\alpha] = f_*[\beta]$ and $g_*[\alpha] = g_*[\beta]$, which means that the path $f \circ \alpha$ is homotopic to $f \circ \beta$ in $\mathcal{X}$ (fixing basepoint x_0), and also that the path $g \circ \alpha$ is homotopic to $g \circ \beta$ in $\mathcal{Y}$ (fixing basepoint y_0). So, let $\Phi = \{\phi_t\}_{t \in [0,1]}$ be a basepoint-fixing homotopy of $f \circ \alpha$ into $f \circ \beta$ in $\mathcal{X}$, and let $\Gamma = \{\gamma_t\}_{t \in [0,1]}$ be a basepoint-fixing homotopy of $f \circ \alpha$ into $f \circ \beta$ in $\mathcal{Y}$. Thus, for each $t \in [0, 1]$, $\phi_t : [0, 1] \longrightarrow \mathcal{X}$ is a loop based at x_0 and $\gamma_t : [0, 1] \longrightarrow \mathcal{Y}$ is a loop based at y_0.

For all $t \in [0, 1]$, define $\theta_t : [0, 1] \longrightarrow \mathcal{Z}$ by $\theta_t(s) = (\phi_t(s), \gamma_t(s))$. Then θ_t is a loop in $\mathcal{Z}$ based at z_0. Furthermore, the family $\Theta := \{\theta_t\}_{t \in [0,1]}$ is equivalent to a function $\Theta : [0, 1] \times [0, 1] \longrightarrow \mathcal{Z}$ defined by $\Theta(t, s) = (\Phi(t, s), \Gamma(t, s))$, and Θ is continuous because Φ and Γ are continuous (because they are homotopies). Thus, Θ is a basepoint-fixing homotopy from α to β in $\mathcal{Z}$. Thus, $[\alpha] = [\beta]$, as desired.

Surjective: Let $[\alpha] \in \pi_1(\mathcal{X}, x_0)$ and let $[\beta] \in \pi_1(\mathcal{Y}, y_0)$. We want some $[\gamma] \in \pi_1(\mathcal{Z}, z_0)$ such that $h[\gamma] = ([\alpha], [\beta])$. Define $\gamma : [0, 1] \longrightarrow \mathcal{Z}$ by $\gamma(t) = (\alpha(t), \beta(t))$. Then clearly $f_*[\gamma] = [f \circ \gamma] = [\alpha]$ and $g_*[\gamma] = [g \circ \gamma] = [\beta]$, so that $h[\gamma] = ([\alpha], [\beta])$, as desired.

□

2. The *Sorgenfrey line* is the set $\mathbb{R}$ of real numbers equipped with the topology $\mathfrak{T}$ generated by all half-open intervals $[a, b)$, for any $a, b \in \mathbb{R}$ (this is sometimes called the *half-open interval topology*).

   $(\frac{10}{100})$ (a) Show that $(\mathbb{R}, \mathfrak{T})$ is Hausdorff.

   Solution: Let $x, y \in \mathbb{R}$. Assume $x < y$. Let $b := \frac{x+y}{2}$. Then $x < b < y$. Let $a < x$ and let $c > y$, and define $\mathcal{U} := [a, b)$ and $\mathcal{V} := [b, c)$. Then $x \in \mathcal{U}$ and $y \in \mathcal{V}$ and $\mathcal{U}$ and $\mathcal{V}$ are disjoint open sets in the Sorgenfrey topology; hence $(\mathbb{R}, \mathfrak{T})$ is Hausdorff. □

   $(\frac{5}{100})$ (b) Show that, for any $a < b$, the interval $[a, b)$ is both closed and open in the Sorgenfrey topology.

   Solution: $[a, b)$ is $\mathfrak{T}$-open by definition. To show that $[a, b)$ is closed, we must show that $[a, b)^{\complement}$ is open. But $[a, b)^{\complement} = (-\infty, a) \sqcup [b, \infty)$, and

$$(-\infty, a) \quad = \quad \bigcup_{c<a} [c, a) \quad \text{and} \quad (-\infty, a) \quad = \quad \bigcup_{c>b} [b, c)$$

are both unions of Sorgenfrey basis intervals, hence unions of open sets, hence open. Thus, $(-\infty, a) \sqcup [b, \infty)$ is open; thus $[a, b)$ is closed. □

$(\frac{5}{100})$ (c) Conclude that $(\mathbb{R}, \mathfrak{T})$ is *totally disconnected* —ie. for any distinct $x, y \in \mathbb{R}$, there is a clopen set $\mathcal{S}$ with $x \in \mathcal{S}$ and $y \notin \mathcal{S}$.

Solution: Suppose $x < y$ and let $b = \frac{x+y}{2}$ and $a < x$ as in part (a). Then $\mathcal{S} = [a, b)$ is a clopen set (by (b)) which contains x but not y. □

3. Let $\mathcal{X}$ be a Banach space. Let $L : \mathcal{X} \longrightarrow \mathcal{X}$ be a bounded linear operator with $\|L\| < 1$, and let $u \in \mathcal{X}$. Consider the function $\Phi : \mathcal{X} \longrightarrow \mathcal{X}$ defined by $\Phi(x) = L(x) - u$.

$(\frac{5}{100})$ (a) Show that Φ is a contraction.

Solution: For any $x, y \in \mathcal{X}$, $\Phi(x) - \Phi(y) = L(x) + u - L(y) - u = L(x) - L(y) = L(x - y)$. Thus

$$\|\Phi(x) - \Phi(y)\| = \|L(x - y)\| \leq \|L\| \cdot \|x - y\| = C\|x - y\|.$$

where $C := \|L\| < 1$ is the contraction constant. □

$(\frac{5}{100})$ (b) Show that there exists a unique $x \in \mathcal{X}$ such that $x = L(x) + u$.

Solution: $x = L(x) + u$ iff $\Phi(x) = x$ —i.e. x is a Φ-fixed point. However, Φ is a contraction and $\mathcal{X}$ is a complete metric space, so the Contraction Mapping Theorem says that Φ has a unique Φ-fixed point. □

4. A *Polish space* is a metrizable topological space $(\mathcal{X}, \mathfrak{T})$ such that there exists *some* metric
$(\frac{10}{100})$ d on $\mathcal{X}$ (compatible with $\mathfrak{T}$) such that $(\mathcal{X}, d)$ is a complete metric space.

Let $(\mathcal{X}, \mathfrak{T})$ be a Polish space and let $\mathcal{Y} \subset \mathcal{X}$ be a closed subspace; show that $\mathcal{Y}$ is also Polish.

Solution: Let d_X be the complete metric on $\mathcal{X}$. Then $\mathcal{Y}$ is a closed subset of $(\mathcal{X}, d_X)$; hence if $d_\mathcal{Y}$ is the restriction of $d_\mathcal{X}$ to $\mathcal{Y}$, then $(\mathcal{Y}, d_\mathcal{Y})$ is also a complete metric space. Thus, $\mathcal{Y}$ admits a complete metrization, so $\mathcal{Y}$ is Polish. □

5. Let $\mathcal{X}$ be an infinite set and let $\mathfrak{T}$ be the *cofinite topology:*

$$\mathfrak{T} := \{\mathcal{O} \subset \mathcal{X} \; ; \; \mathcal{O}^\complement \text{ is finite}\}.$$

$(\frac{8}{100})$ (a) Show that *every* subset of $(\mathcal{X}, \mathfrak{T})$ is compact.

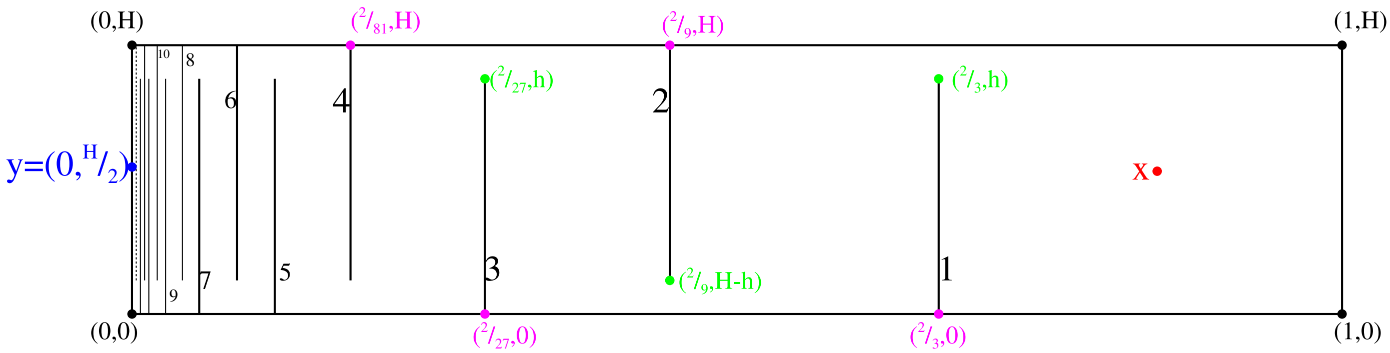
Solution: Let $\mathcal{S} \subset \mathcal{X}$. Let $\{\mathcal{O}_i\}_{i \in \mathbb{I}}$ be an open cover of $\mathcal{S}$. Fix $i_0 \in \mathbb{I}$; then $\mathcal{S} \setminus \mathcal{O}_{i_0} \subset \mathcal{O}_{i_0}^\complement$ is finite (because $\mathcal{O}_{i_0}^\complement$ is finite). Suppose $\mathcal{S} \setminus \mathcal{O}_{i_0} = \{s_1, s_2, \ldots, s_N\}$. For each $n \in [1...N]$, there is some $i_n \in \mathbb{I}$ such that $s_n \in \mathcal{O}_{i_n}$. Thus, the collection $\{\mathcal{O}_{i_n}\}_{n=0}^N$ covers $\mathcal{S}$, because

$$\bigcup_{n=0}^N \mathcal{O}_{i_n} = \mathcal{O}_{i_0} \cup \bigcup_{n=1}^N \mathcal{O}_{i_n} \supseteq \mathcal{O}_{i_0} \cup \{s_1, \ldots, s_N\} \supseteq \mathcal{S}.$$

Thus, $\{\mathcal{O}_{i_n}\}_{n=0}^N$ is a finite subcover, as desired. □

$(\frac{2}{100})$ (b) Which subsets of $\mathcal{X}$ are closed?

Solution: $\mathcal{S}$ is closed $\iff$ $\mathcal{S}^\complement$ is open $\iff$ $\mathcal{S}^\complement$ is cofinite $\iff$ $\mathcal{S}$ is finite. □



6. Consider the ***triadic labyrinth*** shown above. We begin with a box $\mathbf{B} := [0,1] \times [0,H]$ (for some $H > 0$). We remove the line segment '1' which starts at the point $(\frac{2}{3}, 0)$ and goes up to $(\frac{2}{3}, h)$ (where $\frac{h}{2} < h < H$). Then we remove line segment '2' which starts at $(\frac{2}{9}, H)$ and goes down to $(\frac{2}{9}, H-h)$. Then we remove line segement '3' which starts at $(\frac{2}{27}, 0)$, and line segment '4' starting at $(\frac{2}{81}, H)$, and so on. The nth line segment extends upwards from $(\frac{2}{3^n}, 0)$ if n is odd, or downwards from $(\frac{2}{3^n}, H)$ if n is even, and has length h. The ***Labyrinth*** is the set $\mathbf{L}$ which remains when infinitely many line segments have been

$(\frac{15}{100})$ removed from $\mathbf{B}$

Show that $\mathbf{B}$ is *not* path connected.

(**Hint:** *Consider the point* $\mathbf{y} = (0, \frac{H}{2})$ *on the left boundary of* $\mathbf{L}$, *and the point* $\mathbf{x}$ *in* $\mathbf{L}$, *shown in the picture. Show that it is impossible to connect* $\mathbf{x}$ *to* $\mathbf{y}$ *with a continuous path in* $\mathbf{L}$.)

Solution: Suppose $\gamma : [0,1] \longrightarrow \mathbf{L}$ was a continous path from $\mathbf{x}$ to $\mathbf{y}$. Thus, $\gamma(1) = \mathbf{y}$. Thus, $\lim_{t \to 1} \gamma(t) = \mathbf{y}$, because γ is continuous. Thus, if $\gamma(t) = (\gamma_1(t), \gamma_2(t))$, then $\lim_{t \to 1} \gamma_1(t) = 0$ and $\lim_{t \to 1} \gamma_1(t) = \dfrac{H}{2}$. However, as $t \to 1$, $\gamma_2(t)$ must swing back and forth infinitely often between h and $H - h$ (because γ must avoid the line segments $1, 2, 3, 4, \ldots$). Thus, we can find a sequence of times $\{t_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} t_n = 1$ and $\lim_{n \to \infty} \gamma_2(t_n) = h$, and another sequence of times $\{s_n\}_{n=1}^{\infty}$ with $\lim_{n \to \infty} s_n = 1$ and $\lim_{n \to \infty} \gamma_2(s_n) = H - h$. But then $\lim_{t \to 1} \gamma(t) = h$ and $\lim_{t \to 1} \gamma(t) = H - h$, so we must have $h = \frac{H}{2} = H - h$. But $h > \frac{H}{2}$, so this is a contradiction.

By contradiction, the path γ cannot exist. Thus, $\mathbf{x}$ cannot be connected to $\mathbf{y}$. Thus, $\mathbf{L}$ is not path-connected. □

$(\frac{10}{100})$ 7. Let $\mathcal{X}$ be a *discrete* topological space. Show that $\Big(\mathcal{X}$ is compact$\Big) \iff \Big(\mathcal{X}$ is finite$\Big)$.

Solution: "$\Longrightarrow$" For all $x \in \mathcal{X}$, let $\mathcal{O}_x := \{x\}$. Then $\mathcal{O}_x$ is an open set (because $\mathcal{X}$ is discrete) which contains x, and $\{\mathcal{O}_x\}_{x \in \mathcal{X}}$ is an open cover of $\mathcal{X}$ which has *no* proper subcover. Thus,

$$\Big(\mathcal{X} \text{ is compact}\Big) \iff \left(\begin{array}{l}\text{Every open cover} \\ \text{has a finite subcover}\end{array}\right) \Longrightarrow \Big(\{\mathcal{O}_x\}_{x \in \mathcal{X}} \text{ is finite}\Big) \iff \Big(\mathcal{X} \text{ is finite}\Big).$$

"$\Longleftarrow$" Suppose $\mathcal{X}$ is finite, and let $\mathcal{X} = \{x_1, \ldots, x_n\}$. Let $\{\mathcal{O}_i\}_{i \in \mathbb{I}}$ be any open cover of $\mathcal{X}$. For each $n \in [1..N]$, let $i_n \in \mathbb{I}$ be such that $x_n \in \mathcal{O}_{i_n}$. Then $\{\mathcal{O}_{i_1}, \ldots, \mathcal{O}_{i_n}\}$ is a finite subcover. This works for any open cover, so $\mathcal{X}$ is compact. □