Math 332 — Groups & Symmetry, Final Exam — April 16, 2008

1. Let $(\mathcal{A}, +)$ be an abelian group with identity element 0. Let $n \in \mathbb{N}$. Let $\mathcal{A}[n] := \{a \in \mathcal{A} \ ; \ n\,a = 0\}$, where $n\,a := \underbrace{a + a + \cdots + a}_{n \text{ times}}$. Show that $\mathcal{A}[n]$ is a subgroup of $\mathcal{A}$.

Solution: *Closed under addition.* Suppose $a, b \in \mathcal{A}[n]$. Then

$$n(a+b) \;=\; \underbrace{(a+b) + \cdots + (a+b)}_{n \text{ times}} \;\underset{(*)}{=}\; \underbrace{a + \cdots + a}_{n \text{ times}} + \underbrace{b + \cdots + b}_{n \text{ times}} \;=\; n\,a + n\,b \;=\; 0 + 0 \;=\; 0.$$

Here, $(*)$ is because '$+$' is commutative. Thus, $(a+b) \in \mathcal{A}[n]$.

Closed under inversion. Suppose $a \in \mathcal{A}[n]$. Then

$$0 \;=\; n\,0 \;=\; n(a-a) \;\underset{(*)}{=}\; \underbrace{a + \cdots + a}_{n \text{ times}} - \underbrace{a - \cdots - a}_{n \text{ times}} \;=\; 0 - \underbrace{a - \cdots - a}_{n \text{ times}} \;=\; n\,(-a).$$

Here, $(*)$ is because '$+$' is commutative. Thus, $(-a) \in \mathcal{A}[n]$. □

2. Let $(\mathcal{G}, \cdot)$ be a group with identity e. Let $\mathcal{K}, \mathcal{L} \lhd \mathcal{G}$ be two normal subgroups, with $\mathcal{K} \cap \mathcal{L} = \{e\}$ and $\mathcal{K} \vee \mathcal{L} = \mathcal{G}$. Show that $\mathcal{G}/\mathcal{L} \cong \mathcal{K}$ and $\mathcal{G}/\mathcal{K} \cong \mathcal{L}$.

Solution: Note that $\mathcal{K} \vee \mathcal{L} = \mathcal{K}\mathcal{L}$ by Lemma 34.4 (p.308) because both $\mathcal{K}$ add $\mathcal{L}$ are normal. Thus, $\mathcal{G} := \mathcal{K}\mathcal{L}$. Thus,

$$\frac{\mathcal{G}}{\mathcal{L}} \;=\; \frac{\mathcal{K}\mathcal{L}}{\mathcal{L}} \;\overset{(\diamond)}{\cong}\; \frac{\mathcal{K}}{\mathcal{K} \cap \mathcal{L}} \;=\; \frac{\mathcal{K}}{\{e\}} \;\cong\; \mathcal{K},$$

where $(\diamond)$ is the Diamond Isomorphism Theorem (Thm 34.5, p.308). The proof that $\mathcal{G}/\mathcal{K} \cong \mathcal{L}$ is analogous: just switch $\mathcal{K}$ and $\mathcal{L}$. □

3. Let $\phi : \mathcal{G} \longrightarrow \mathcal{H}$ be a group homomorphism. Let $\mathcal{N} \lhd \mathcal{H}$ be a normal subgroup. Show that $\mathcal{M} := \phi^{-1}[\mathcal{N}]$ is a normal subgroup of $\mathcal{G}$.

Solution: Let $g \in \mathcal{G}$ and let $m \in \mathcal{M}$. We must show that $gmg^{-1} \in \mathcal{M}$. Let $h := \phi(g) \in \mathcal{H}$ and $n := \phi(m) \in \mathcal{N}$. Then $\phi(gmg^{-1}) = hnh^{-1} \in \mathcal{N}$, because $\mathcal{N} \lhd \mathcal{H}$; hence $gmg^{-1} \in \phi^{-1}(\mathcal{N}) = \mathcal{M}$, as desired.

Thus, $gmg^{-1} \in \mathcal{M}$ for all $g \in \mathcal{G}$ and let $m \in \mathcal{M}$. Thus, $\mathcal{M} \lhd \mathcal{G}$. □

4. Let $(\mathcal{G}, \cdot)$ be a finite group with with identity e. Let $n := |\mathcal{G}|$. Let $g \in \mathcal{G}$, and let $k := |g| = \min\{m > 0 \ ; \ g^m = e\}$. Show that k divides n.

Solution: Let $\mathcal{H} \preceq \mathcal{G}$ be the cyclic subgroup generated by g. Then $k = |\mathcal{H}|$ (by definition; see p.59). But $|\mathcal{H}|$ divides $|\mathcal{G}|$ by Lagrange's theorem (Thm. 10.10, p.100). Thus, $k|n$. □

5. Let $\mathbb{R}^{n\times n}$ be the set of all $n \times n$ real matrices. $\mathbb{GL}^n := \{\mathbf{A} \in \mathbb{R}^{n\times n} \ ; \ \mathbf{A} \text{ is invertible}\}$; then $\mathbb{GL}^n$ is a group under matrix multiplication (the *general linear group*), whose identity element is the identity matrix.

(a) Let $\sigma : [1...N] \longrightarrow [1...N]$ be a permutation. Define $\mathbf{A}_\sigma := \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix}$ by

$$a_{nm} \quad := \quad \begin{cases} 1 & \text{if } m = \sigma(n) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Let $\mathbf{x} := (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. Compute $\mathbf{A}_\sigma \mathbf{x}$.

Solution: $\mathbf{A}_\sigma \mathbf{x} = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$. To see thus, let $\mathbf{A}_\sigma \mathbf{x} = (y_1, y_2, \ldots, y_n)$. Then for all $i \in [1...n]$,

$$y_i \quad = \quad \sum_{j=1}^{n} a_{ij}\, x_j \underset{(*)}{=} x_{\sigma(i)}.$$

Here $(*)$ is by eqn.(1). □

(b) Define $\phi : \mathbf{S}_n \longrightarrow \mathbb{GL}^n$ by $\phi(\sigma) = \mathbf{A}_\sigma$. Let $\mathcal{S} := \phi[\mathbf{S}_n] \subseteq \mathfrak{L}$. Show that $\phi : \mathbf{S}_n \longrightarrow \mathcal{S}$ is an group isomorphism.

Solution: $\phi : \mathbf{S}_n \longrightarrow \mathcal{S}$ is surjective because $\mathcal{S} := \phi[\mathbf{S}_n]$ by definition. We must show that ϕ is an injective homomorphism.

Homomorphism: Let $\sigma, \tau \in \mathbf{S}_n$. We must show $\phi(\sigma\tau) = \phi(\sigma)\phi(\tau)$. In other words, we must show that $\mathbf{A}_{\sigma\tau} = \mathbf{A}_\sigma \mathbf{A}_\tau$. To do this, it suffices to show that $\mathbf{A}_{\sigma\tau}\mathbf{x} = \mathbf{A}_\sigma \mathbf{A}_\tau \mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}^n$. But for all $i \in [1...n]$,

$$(\mathbf{A}_{\sigma\tau}\mathbf{x})_i \underset{(*)}{=} x_{\sigma\tau(i)} \quad = \quad x_{\sigma(\tau(i))} \underset{(*)}{=} (\mathbf{A}_\sigma(\mathbf{A}_\tau \mathbf{x}))_i$$

where both $(*)$ are by part (a). Thus, $\mathbf{A}_{\sigma\tau}\mathbf{x} = \mathbf{A}_\sigma \mathbf{A}_\tau \mathbf{x}$. this works for all $\mathbf{x} \in \mathbb{R}^n$; thus, $\phi(\sigma\tau) = \phi(\sigma)\phi(\tau)$.

Injective: Suppose $\sigma \neq \tau$; we must show $\phi(\sigma) \neq \phi(\tau)$, which means that $\mathbf{A}_\sigma \neq \mathbf{A}_\tau$. It suffices to find some $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{A}_\sigma \mathbf{x} \neq \mathbf{A}_\tau \mathbf{x}$.

Let $\mathbf{x} = (1, 2, \ldots, n)$. Then part (a) implies that $\mathbf{A}_\sigma \mathbf{x} = (\sigma(1), \ldots, \sigma(n))$ and $\mathbf{A}_\tau \mathbf{x} = (\tau(1), \ldots, \tau(n))$. If $\sigma \neq \tau$, then $\sigma(i) \neq \tau(i)$ for at least one $i \in [1...n]$. Thus, $(\sigma(1), \ldots, \sigma(n)) \neq (\tau(1), \ldots, \tau(n))$; thus, $\mathbf{A}_\sigma \mathbf{x} \neq \mathbf{A}_\tau \mathbf{x}$; thus $\phi(\sigma) \neq \phi(\tau)$, as desired. □

(c) Let $\mathcal{G}$ be any group of cardinality n. Show that $\mathcal{G}$ is isomorphic to a subgroup of $\mathbb{GL}^n$.

Solution: Let $\mathbf{S}_n$ be the group of permutations of $[1...n]$. Then $\mathcal{G}$ is isomorphic to some subgroup $\mathcal{H} \preceq \mathbf{S}_n$, by Cayley's Theorem (Theorem 8.16) . If $\phi : \mathbf{S}_n \longrightarrow \mathbb{GL}^n$ is the isomorphism from (b), and $\mathcal{I} := \phi[\mathcal{H}] \subset \mathbb{GL}^n$, then $\mathcal{I}$ is a subgroup of $\mathbb{GL}^n$ and $\mathcal{H} \cong \mathcal{I}$. Thus, we have $\mathcal{G} \cong \mathcal{H} \cong \mathcal{I}$; hence $\mathcal{G} \cong \mathcal{I}$, as desired. □