Math 220 — Final Exam —December 19, 2005.

- *Justify your answers.* Show all steps in your computations. • Do *not* use a calculator.
- Please indicate your final answer by putting a $\boxed{\text{box}}$ around it (if appropriate).
- Please write neatly and legibly. • If you have any questions, please ask.

Part I. *Multiple Choice:*

1. $(\frac{5}{200})$ Let $\mathcal{P}$, $\mathcal{Q}$ and $\mathcal{R}$ be propositions. Which of the following is the *logical negation* of the statement "$\mathcal{P}$ and $\mathcal{Q}$ and $\mathcal{R}$"?

   (a) $\mathcal{P}$ or $\mathcal{Q}$ or $\mathcal{R}$.

   (b) not($\mathcal{P}$) and not($\mathcal{Q}$) and not($\mathcal{R}$).

   (c) not($\mathcal{P}$) or not($\mathcal{Q}$) or not($\mathcal{R}$).

   (d) not($\mathcal{P}$) and [not($\mathcal{Q}$) or not($\mathcal{R}$)].

   (e) None of the above.

   (f) All of the above.

2. $(\frac{5}{200})$ Let $\mathcal{P}(x)$ and $\mathcal{Q}(x)$ be statements about some object x. Let $\mathcal{R}$ be the statement "For all x, $\mathcal{P}(x)$ implies $\mathcal{Q}(x)$". Which of the following is the *logical negation* of $\mathcal{R}$?

   (a) For all x, $\mathcal{Q}(x)$ implies $\mathcal{P}(x)$.

   (b) For all x, $\mathcal{P}(x)$ and not$\mathcal{Q}(x)$.

   (c) For all x, not$\mathcal{P}(x)$ implies not$\mathcal{Q}(x)$.

   (d) For all x, not$\mathcal{Q}(x)$ implies not$\mathcal{P}(x)$.

   (e) There exists x such that $\mathcal{Q}(x)$ implies $\mathcal{P}(x)$.

   (f) There exists x such that $\mathcal{P}(x)$ and not$\mathcal{Q}(x)$.

   (g) There exists x such that not$\mathcal{P}(x)$ implies not$\mathcal{Q}(x)$.

   (h) There exists x such that not$\mathcal{Q}(x)$ implies not$\mathcal{P}(x)$.

   (i) None of the above.

   (j) All of the above.

3. Let $\mathcal{X}$ and $\mathcal{Y}$ be sets, and let $f : \mathcal{X} \longrightarrow \mathcal{Y}$ be a function.

   $(\frac{5}{200})$ **(i)** Which of the following statements is *equivalent* to the statement "f is injective"?

   $(\frac{5}{200})$ **(ii)** Which of the following statements is *equivalent* to the statement "f is surjective"?

   $(\frac{5}{200})$ **(iii)** Which of the following statements is *equivalent* to the statement "f is bijective"?

   (a) For all $x \in \mathcal{X}$, there exists $y \in \mathcal{Y}$ such that $f(x) = y$.

   (b) For all $y \in \mathcal{Y}$, there exists $x \in \mathcal{X}$ such that $f(x) = y$.

(c) For all $x \in \mathcal{X}$, there exists a *unique* $y \in \mathcal{Y}$ such that $f(x) = y$.

(d) For all $y \in \mathcal{Y}$, there exists a *unique* $x \in \mathcal{X}$ such that $f(x) = y$.

(e) For all $x, x' \in \mathcal{X}$, if $x = x'$, then $f(x) = f(x')$.

(f) For all $x, x' \in \mathcal{X}$, if $f(x) = f(x')$, then $x = x'$.

(g) For all $x, x' \in \mathcal{X}$, $f(x) = f(x')$ if and only if $x = x'$.

(h) There exists $y \in \mathcal{Y}$, such that for all $x \in \mathcal{X}$, $f(x) = y$.

(i) There exists $x \in \mathcal{X}$, such that for all $y \in \mathcal{Y}$, $f(x) = y$.

(j) There exists a *unique* $y \in \mathcal{Y}$, such that for all $x \in \mathcal{X}$, $f(x) = y$.

(k) There exists a *unique* $x \in \mathcal{X}$, such that for all $y \in \mathcal{Y}$, $f(x) = y$.

(l) None of the above.

(m) All of the above.

Part II. *Long answer:*

$(\frac{5}{200})$ 4. (a) Show that, for any $n \in \mathbb{N}$, $10^n \underset{3}{\equiv} 1$.

$(\frac{20}{200})$ (b) Suppose $B \in \mathbb{N}$ has decimal expansion $B = b_k b_{k-1} \ldots b_2 b_1 b_0$ (where $0 \leq b_i \leq 9$ for all i). Show that

$$\Big(B \text{ is divisible by } 3\Big) \iff \Big(b_0 + b_1 + b_2 + \cdots + b_k \text{ is divisible by } 3\Big).$$

(For example $B = 217$ is *not* divisible by 3, because $2 + 1 + 7 = 10$ is *not* divisible by 3. However, $B = 6123$ *is* divisible by 3, because $6 + 1 + 2 + 3 = 12$ *is* divisible by 3)

5. Let $\mathbf{X}$ be a set, and let $\mathcal{P}(\mathbf{X}) := \{\mathbf{A} \subseteq \mathbf{X}\}$ be the *power set* of $\mathbf{X}$. We define the *symmetric difference* operator $\triangle$ on $\mathcal{P}(\mathbf{X})$ as follows: For any $\mathbf{A}, \mathbf{B} \subseteq \mathbf{X}$,

$$\mathbf{A}\triangle\mathbf{B} \quad := \quad (\mathbf{A} \cup \mathbf{B}) \setminus (\mathbf{A} \cap \mathbf{B}) \quad = \quad \{x \in \mathbf{X} \,;\, \text{either } x \in \mathbf{A} \text{ or } x \in \mathbf{B}, \text{ but not both}\}.$$

$(\frac{5}{200})$ (a) Show that $\triangle$ is *commutative*. That is, for any $\mathbf{A}, \mathbf{B} \subset \mathbf{X}$, $\mathbf{A}\triangle\mathbf{B} = \mathbf{B}\triangle\mathbf{A}$.

$(\frac{20}{200})$ (b) Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \subset \mathbf{X}$. Draw Venn diagrams of the following twelve sets:

i. $\mathbf{A} \cup \mathbf{B}$, $\mathbf{A} \cap \mathbf{B}$ and $\mathbf{A}\triangle\mathbf{B}$.

ii. $(\mathbf{A}\triangle\mathbf{B}) \cup \mathbf{C}$, $(\mathbf{A}\triangle\mathbf{B}) \cap \mathbf{C}$, and $(\mathbf{A}\triangle\mathbf{B})\triangle\mathbf{C}$.

iii. $\mathbf{B} \cup \mathbf{C}$, $\mathbf{B} \cap \mathbf{C}$ and $\mathbf{B}\triangle\mathbf{C}$.

iv. $\mathbf{A} \cup (\mathbf{B}\triangle\mathbf{C})$, $\mathbf{A} \cap (\mathbf{B}\triangle\mathbf{C})$, and $\mathbf{A}\triangle(\mathbf{B}\triangle\mathbf{C})$.

$(\frac{5}{200})$ (c) Use your diagrams in (b) to show that that $\triangle$ is *associative*.

$(\frac{5}{200})$ (d) Find a set $\mathbf{E} \subseteq \mathbf{X}$ such that, for any $\mathbf{A} \subseteq \mathbf{X}$, $\mathbf{E}\triangle\mathbf{A} = \mathbf{A}$.

$(\frac{5}{200})$ (e) For any set $\mathbf{A} \subseteq \mathbf{X}$, show that there exists a set $\mathbf{B}$ such that $\mathbf{A}\triangle\mathbf{B} = \mathbf{E}$.

$(\frac{5}{200})$ (f) Conclude that $(\mathcal{P}(\mathbf{X}), \triangle)$ is an *abelian group.*

(g) Let $f : \mathbf{X} \longrightarrow \mathbf{Y}$ be any function. Recall that we define the function $\overleftarrow{f} : \mathcal{P}(\mathbf{Y}) \longrightarrow \mathcal{P}(\mathbf{X})$
$(\frac{10}{200})$ by $\overleftarrow{f}[\mathbf{A}] := \{x \in \mathbf{X}\ ;\ f(x) \in \mathbf{A}\}$ for any subset $\mathbf{A} \subseteq \mathbf{Y}$.

Show that $\overleftarrow{f}$ is a *group homomorphism* from the group $(\mathcal{P}(\mathbf{Y}), \triangle)$ to the group $(\mathcal{P}(\mathbf{X}), \triangle)$.

6. Rudolph owes Nicholas \$134.00. However, Rudolph only has \$7.00 bills in his wallet,
$(\frac{20}{200})$ and Nicholas only has threenies (that is, \$3.00 coins). Is it possible for Rudolph to pay Nicholas the debt through an exchange of bills and threenies? If so, then *how*? If not, then *why* not?

7. The buffet at the Winter Solstice party has 7 different kinds of dessert.

(a) Suppose you want to put three different desserts on your plate. How many different
$(\frac{5}{200})$ ways can you do this?

(b) Suppose your three friends Alice, Bob, and Carol ask you to get one dessert for each of them. Thus, you want to put one dessert on each of their plates. How many ways
$(\frac{5}{200})$ are there to do this?

(c) Suppose that there is *only one* serving left of each of the seven desserts in the buffet.
$(\frac{5}{200})$ Now how many ways can you give one dessert to each of Alice, Bob, and Carol?

8. Let $\mathcal{P} := \{p_1, p_2, p_3, \ldots, p_n\}$ be a *finite* set of prime numbers.

$(\frac{10}{200})$ (a) Construct an integer m which is not divisible by *any* of the primes in $\mathcal{P}$.

$(\frac{10}{200})$ (b) Explain why this implies that there exists a prime number *not* in $\mathcal{P}$.

$(\frac{15}{200})$ (c) Conclude that the set of prime numbers must be *infinite*.

9. Let $\mathcal{R} := \{\,{}_1r,\ {}_2r,\ {}_3r,\ {}_4r, \ldots\}$ be a denumerable set of real numbers in the interval $[0, 1]$. Suppose ${}_kr$ has decimal expansion ${}_ir = 0.\ {}_ir_1\ {}_ir_2\ {}_ir_3\ {}_ir_4\ {}_ir_5 \ldots$ (where ${}_ir_j \in \{0, 1, \ldots, 9\}$ for each $i, j \in \mathbb{N}$)

Suppose we write these numbers in a table as follows:

$$\begin{array}{cccccccc}
0 & . & {}_1r_1 & {}_1r_2 & {}_1r_3 & {}_1r_4 & {}_1r_5 & \cdots \\
0 & . & {}_2r_1 & {}_2r_2 & {}_2r_3 & {}_2r_4 & {}_2r_5 & \cdots \\
0 & . & {}_3r_1 & {}_3r_2 & {}_3r_3 & {}_3r_4 & {}_3r_5 & \cdots \\
0 & . & {}_4r_1 & {}_4r_2 & {}_4r_3 & {}_4r_4 & {}_4r_5 & \cdots \\
0 & . & {}_5r_1 & {}_5r_2 & {}_5r_3 & {}_5r_4 & {}_5r_5 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

(a) Construct a number $s \in [0, 1]$ which is *not* an element of $\mathcal{R}$. (**Hint:** *Specify the decimal*
$(\frac{10}{200})$ *expansion of* s.)

$(\frac{15}{200})$ (b) Use this to argue that the set $[0, 1]$ is *not* denumerable.