Math 220 (*Intro. to Pure Math.*) — Final Exam 2004-12-13

1. Let $(\mathcal{G}, *)$ be a group (with identity e).

$(\frac{15}{200})$ (a) Show that the inverse of any element in $\mathcal{G}$ is *unique*. In other words, let $g \in \mathcal{G}$, and suppose a and b are two elements satisfying the definition of 'inverse of g':

$$a * g = e = g * a; \quad \text{and} \quad b * g = e = g * b.$$

Conclude that $a = b$.

Solution: $a \underset{(1)}{=} a * e \underset{(2)}{=} a * (g * b) \underset{(3)}{=} (a * g) * b \underset{(4)}{=} e * b \underset{(1)}{=} b$.

(1) because e is identity. (2) because b is an inverse of g. (3) because multiplication is associative. (4) because a is an inverse of g. ______ □

$(\frac{15}{200})$ (b) Let $(\mathcal{H}, \star)$ be another group, with identity ϵ. and let $\phi : \mathcal{G} \longrightarrow \mathcal{H}$ be a group isomorphism. Let g^{-1} be the inverse of g, and let $h = \phi(g)$. Show that $\phi(g^{-1}) = h^{-1}$.

(**Hint:** *You may assume $\phi(e) = \epsilon$.*)

Solution: Let $k = \phi(g^{-1})$. Then $k \star h = \phi(g^{-1}) \star \phi(g) \underset{(*)}{=} \phi(g^{-1} * g) \underset{(\dagger)}{=} \phi(e) \underset{(\ddagger)}{=} \epsilon$.

Here $(*)$ is by definition of isomorphism. $(\dagger)$ is because g^{-1} is the inverse of g, and $(\ddagger)$ is by the hint.

Thus, k satisfies the definining property of the inverse for h. But (a) says there is a *unique* element satisfying this property —namely the (unique) inverse h^{-1} of h. Hence $\phi(k) = h^{-1}$, as desired. ______ □

$(\frac{20}{200})$ 2. Let $P(x) = x^2 + p_1 x + p_0$ be a quadratic polynomial, where $p_0, p_1, p_2 \in \mathbb{R}$. Suppose that $P(x)$ has complex-valued roots $r_1, r_2 \in \mathbb{C}$ (ie. $P(r_1) = P(r_2) = 0$), and suppose that r_1 and r_2 are *not* real (ie. they have nontrivial imaginary part). Show that r_2 must be the *complex conjugate* of r_1. In other words, if $r_1 = x + \mathbf{i}y$, then $r_2 = x - \mathbf{i}y$.

Solution: If r_1 and r_2 are the roots of $P(z)$, then

$$x^2 + p_1 x + p_0 \quad = \quad P(z) \quad \underset{(*)}{=} \quad (z - r_1) \cdot (z - r_2) \quad = \quad z^2 - (r_1 + r_2)z + r_1 \cdot r_2.$$

where $(*)$ is by Descartes' Factor Theorem.

Thus, if $r_1 = x_1 + \mathbf{i}y_1$ and $r_2 = x_2 + \mathbf{i}y_2$, then we have the equations

$$\begin{aligned} p_1 &= -(r_1 + r_2) = -(x_1 + x_2) - \mathbf{i}(y_1 + y_2), \\ \text{and } p_0 &= r_1 \cdot r_2 = (x_1 x_2 - y_1 y_2) + \mathbf{i}(x_1 y_2 + y_1 x_2). \end{aligned}$$

But p_1 and p_0 are *real* numbers, by hypothesis; hence we must have

$$(\mathbf{a}) \quad (y_1 + y_2) = 0 \quad \text{and} \quad (\mathbf{b}) \quad (x_1 y_2 + y_1 x_2) = 0.$$

Equation **(a)** tells us that $y_2 = -y_1$. Substituting this into **(b)** we get

$$0 \quad = \quad x_1 y_2 + y_1 x_2 \quad = \quad -x_1 y_1 + y_1 x_2 \quad = \quad y_1(x_2 - x_1).$$

Hence, if $y_1 \neq 0$, then we must conclude that $x_2 = x_1$. Hence $r_1 = x_1 + \mathbf{i}y_1$ and $r_2 = x_1 - \mathbf{i}y_1 = \overline{r}_1$.
□

3. (a) Let $\mathcal{U} \subset \mathbb{R}^n$ and $\mathcal{V} \subset \mathbb{R}^m$. Let $\phi : \mathcal{U} \longrightarrow \mathcal{V}$. Define what it means for ϕ to be **continuous** on $\mathcal{U}$. $\left(\frac{5}{200}\right)$

Solution: ϕ is *continuous* on $\mathcal{U}$ if the following is true: For any sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{U}$, and any $u \in \mathcal{U}$, $\left(\lim_{n\to\infty} x_n = u \text{ in } \mathcal{U}\right) \Longrightarrow \left(\lim_{n\to\infty} \phi(x_n) = \phi(u) \text{ in } \mathcal{V}.\right)$ ☐

$\left(\frac{20}{200}\right)$ (b) Let $\mathcal{U} \subset \mathbb{R}^n$, $\mathcal{V} \subset \mathbb{R}^m$, and $\mathcal{W} \subset \mathbb{R}^\ell$. Let $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ and $\psi : \mathcal{V} \longrightarrow \mathcal{W}$. Show that

$$\left(\phi \text{ is continuous on } \mathcal{U}, \text{ and } \psi \text{ is continuous on } \mathcal{V}\right)$$
$$\Longrightarrow \left(\psi \circ \phi : \mathcal{U} \longrightarrow \mathcal{W} \text{ is continuous on } \mathcal{U}\right).$$

Solution: Let $u \in \mathcal{U}$ and let $v = \phi(u)$. Let $w = \psi \circ \phi(u) \in \mathcal{W}$. Suppose $\{x_n\}_{n=1}^{\infty} \subset \mathcal{U}$ is sequence such that $\lim_{n\to\infty} x_n = u$. For all $n \in \mathbb{N}$, let $z_n := \psi \circ \phi(x_n)$; we claim that $\lim_{n\to\infty} z_n = w$.

To see this, note that $w = \psi(v)$. For all $n \in \mathbb{N}$, let $y_n := \phi(x_n)$; then $z_n = \psi(y_n)$. Now, ϕ is continuous and $\lim_{n\to\infty} x_n = u$, thus $\lim_{n\to\infty} y_n = v$. But ψ is also continuous; hence $\lim_{n\to\infty} z_n = \phi(v) = w$, as desired. ☐

$\left(\frac{5}{200}\right)$ (c) Define **homeomorphism.**

Solution: $\phi : \mathcal{U} \longrightarrow \mathcal{V}$ is a **homeomorphism** from $\mathcal{U}$ to $\mathcal{V}$ if:

[i] ϕ is bijective (therefore invertible).

[ii] ϕ is continuous on $\mathcal{U}$.

[iii] $\phi^{-1} : \mathcal{V} \longrightarrow \mathcal{U}$ is continuous on $\mathcal{V}$. ☐

$\left(\frac{20}{200}\right)$ (d) Let $\mathcal{U} \subset \mathbb{R}^n$, $\mathcal{V} \subset \mathbb{R}^m$, and $\mathcal{W} \subset \mathbb{R}^\ell$. Show that

$$\left(\phi \text{ is a homeomorphism from } \mathcal{U} \text{ to } \mathcal{V} \text{ and } \psi \text{ is a homeomorphism from } \mathcal{V} \text{ to } \mathcal{W}\right)$$
$$\Longrightarrow \left(\psi \circ \phi \text{ is a homeomorphism from } \mathcal{U} \text{ to } \mathcal{W}\right).$$

Hence, if $\mathcal{U} \cong \mathcal{V}$ and $\mathcal{V} \cong \mathcal{W}$, then $\mathcal{U} \cong \mathcal{W}$.

Solution: We will verify that $\psi \circ \phi$ satisfies the three defining properties of a homeomorphism:

[i] *Bijective:* ϕ is bijective and ψ is bijective, so $\psi \circ \phi$ is bijective by a lemma proved in class.

[ii] *Continuous:* By hypothesis, ϕ is continuous on $\mathcal{U}$ and ψ is continuous on $\mathcal{V}$. Thus, $\psi \circ \phi$ is continuous on $\mathcal{U}$ by part (b).

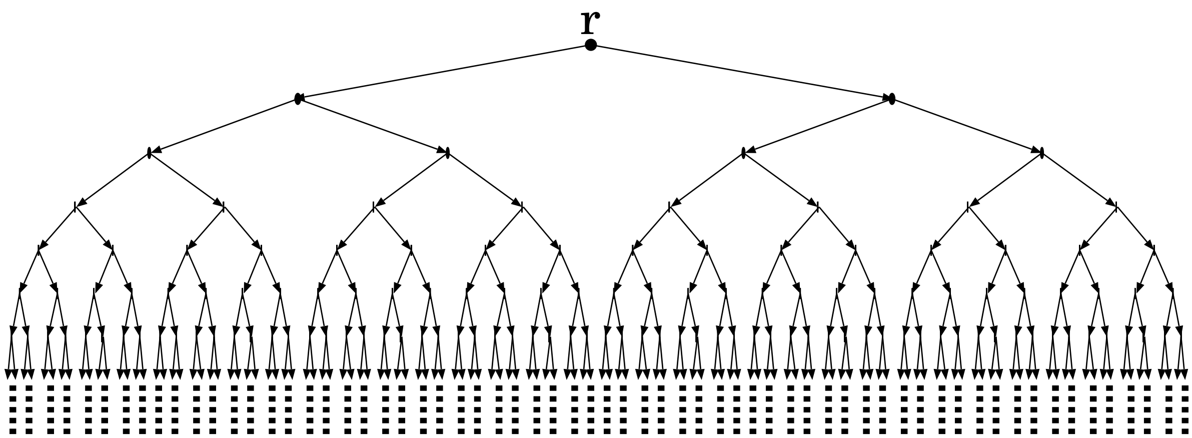
[iii] *Continuous Inverse:* In class, we proved that $(\psi \circ \phi)^{-1} = \phi^{-1} \circ \psi^{-1}$. By hypothesis, $\psi^{-1} : \mathcal{W} \longrightarrow \mathcal{V}$ is continuous on $\mathcal{W}$, and and $\phi^{-1} : \mathcal{V} \longrightarrow \mathcal{U}$ is continuous on $\mathcal{U}$. Thus, $\phi^{-1} \circ \psi^{-1}$ is continuous on $\mathcal{W}$ by part (b). ☐

4. Fix $d \in \mathbb{N}$. If $a \in \mathbb{Z}$, then a *multiplicative inverse* for $a \pmod d$ is a number $b \in \mathbb{Z}$ so that $a \cdot b \equiv 1 \pmod d$. We then say that a is *invertible mod* d. Show that

$\left(\frac{30}{200}\right)$ $$\left(a \text{ is relatively prime to } d\right) \Longrightarrow \left(a \text{ is invertible mod } d\right).$$

Solution: Let $g = \gcd(a, d)$. Recall (p.45-46 in CRM) that we can always find some $b, c \in \mathbb{N}$ such that $g = ba + cd$.

Now, if a and d are relatively prime, then $g = 1$. Thus, we have $ba + cd = 1$. But then, $ba = 1 - cd \equiv 1 \pmod d$. Thus, b is a multiplicative inverse for a. ☐



(In the next question, you may assume the following: $\#(\mathbb{N}) = \#(\mathbb{Z}) = \#(\mathbb{Q}) = \aleph_0$, and $\#([0,1]) = \#([0,1)) = \#(\mathbb{R}) = \#(\mathbb{R}^2) = \#(\mathbb{R}^3) = \mathfrak{c} = 2^{\aleph_0} = 3^{\aleph_0}$.)

5. Consider the infinite binary tree above.

$(\frac{10}{200})$ (a) Fix $n \in \mathbb{N}$. Let $\mathcal{P}_n$ be the set of all descending paths of length n starting at the root node r. Compute $\#(\mathcal{P}_n)$. [For example, $\#(\mathcal{P}_3) = 8$]

Solution: A path of length n involves n distinct choices, each between two alternatives. Each possible sequence of n choices results in a distinct path. Thus, there are 2^n distinct paths of length n.

To put it another way, let $\mathcal{A} = \{0, 1\}$. Then any length-n path corresponds to a sequence $(a_1, a_2, \ldots, a_n)$, where each $a_k \in \mathcal{A}$, and where

$$a_k \quad := \quad \begin{cases} 0 & \text{if} \quad \text{you go } \textit{left} \text{ at level } k. \\ 1 & \text{if} \quad \text{you go } \textit{right} \text{ at level } k. \end{cases}$$

Thus, $\#(\mathcal{P}_n) = \#(\mathcal{A}^n) = \#(\mathcal{A})^n = 2^n$. ______ □

$(\frac{15}{200})$ (b) Let $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 \sqcup \mathcal{P}_3 \sqcup \cdots$ be the set of all descending paths of *any finite length* starting at the root node r. Compute $\#(\mathcal{P})$.

Solution: Let $\mathcal{A} = \{0, 1\}$. Let $\mathcal{A}^*$ be the set of *all finite-length sequences* of 0's and 1's. In other words, $\mathcal{A}^* = \mathcal{A}^1 \sqcup \mathcal{A}^2 \sqcup \mathcal{A}^3 \sqcup \cdots$.

Claim 1: $\#(\mathcal{P}) = \#(\mathcal{A}^*)$.

Proof: As discussed in part (a), any length-n path corresponds to a sequence $(a_1, a_2, \ldots, a_n)$, where each $a_k \in \mathcal{A}$.
This yields a bijection $\phi_n : \mathcal{P}_n \longrightarrow \mathcal{A}^n$. Now we define $\phi : \mathcal{P} \longrightarrow \mathcal{A}^*$ as follows: for any $p \in \mathcal{P}$, $\phi(p) = \phi_n(p)$, if $p \in \mathcal{P}_n$. Then ϕ is a bijection from $\mathcal{P}$ to $\mathcal{A}^*$. Hence $\#(\mathcal{P}) = \#(\mathcal{A}^*)$. □ [Claim 1]

Claim 2: $\#(\mathcal{A}^*) = \aleph_0$.

Proof: There is natural bijection between $\mathcal{A}^*$ and $\mathbb{N}$, because every natural number has a unique (finite length) binary expansion:

$$\begin{array}{rcccccccccc} \mathbb{N}: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \cdots \\ \mathcal{A}^*: & 0 & 1 & 10 & 11 & 100 & 101 & 110 & 111 & 1000 & \cdots \end{array}$$

Thus, $\#(\mathcal{A}^*) = \#(\mathbb{N}) = \aleph_0$. .... □ [Claim 2]

Combining Claims 1 and 2 we get $\#(\mathcal{P}) = \aleph_0$. ________ □

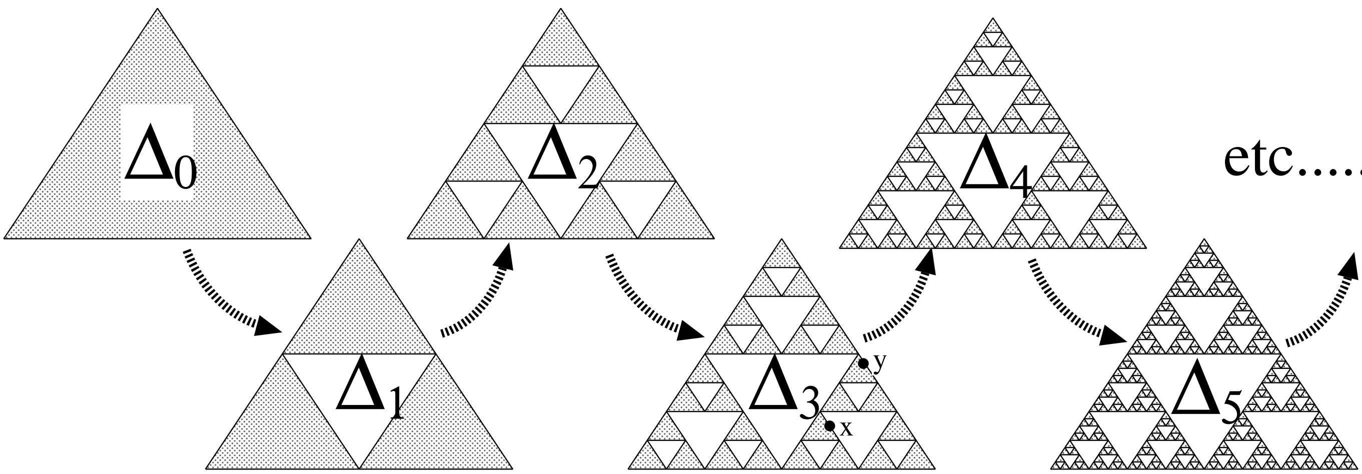
$(\frac{15}{200})$ (c) Let $\mathcal{P}_\infty$ be the set of all descending paths of *infinite* length starting at the root node r. Compute $\#(\mathcal{P}_\infty)$.

Solution: Let $\mathcal{A} = \{0, 1\}$. Let $\mathcal{A}^{\mathbb{N}}$ be the set of *all infinite sequences* of 0's and 1's.

Claim 1: $\#(\mathcal{P}) = \#(\mathcal{A}^{\mathbb{N}})$.

Proof: By the same reasoning as in part (a), any infinite path corresponds to an infinite sequence $(a_1, a_2, \ldots,)$, where each $a_k \in \mathcal{A}$. This yields a bijection $\phi_n : \mathcal{P}_\infty \longrightarrow \mathcal{A}^{\mathbb{N}}$. □ [Claim 1]

But by the hint above, $\#(\mathcal{A}^{\mathbb{N}}) = 2^{\aleph_0} = \mathfrak{c}$. Thus, Claim 1 means that $\#(\mathcal{P}) = \mathfrak{c}$. ____ □



6. Let Δ be the Sierpinski Triangle. That is: $\Delta := \bigcap_{k=0}^{\infty} \Delta_k$, where $\Delta_0 \subset \mathbb{R}^2$ is a triangle, and, for each $k \in \mathbb{N}$, $\Delta_{k+1} \subset \Delta_k$ is the set comprised of the 3^{k+1} triangles which remain after you remove the central triangle from each of the 3^k triangles of Δ_k.

$(\frac{5}{200})$ (a) Consider the two points x and y in Δ_1, shown inside Δ_3 above. What is the smallest set $\mathcal{X}$ you must remove from Δ_3 to disconnect x from y? What is the inductive topological dimension of $\mathcal{X}$?

Solution: Let $\triangle_x$ be the smallest triangle containing x. Observe that $\triangle_x$ can disconnected from the entire remainder of Δ_3 by removing its three corner vertices. In particular, this disconnects x from y. (Alternately, we could disconnect y from x by removing the three corner vertices of $\triangle_y$ instead.)

Either way, we need only remove a set $\mathcal{X}$ containing three points. Since $\mathcal{X}$ is finite, we have $\dim(\mathcal{X}) = 0$. ________ □

$(\frac{25}{200})$ (b) Compute the inductive topological dimension of the Sierpinski Triangle Δ.

Solution: $\dim \Delta = 1$. To see this, let x and y be any two points in Δ.

Claim 1: *There is some n so that x and y are in different 'subtriangles' of Δ_n.*

Proof: Suppose not. Then for every $n \in \mathbb{N}$, we must have x and y in the same 'subtriangle' of Δ_n, which means that $|x - y| \leq \frac{1}{2^n}$. Thus, $|x - y| \leq \frac{1}{2^n}$ for all n, which means that $|x - y| = 0$, which means $x = y$. .... □ [Claim 1]

Then by an argument similar to (a), we can disconnect x from y in Δ_n by removing a set $\mathcal{X}$ from Δ_n containing at most three points. But if x and y are disconnected in $\Delta_n \setminus \mathcal{X}$, then they are certainly disconnected in $\Delta \setminus \mathcal{X}$ (because $\Delta \subset \Delta_n$). Thus, we can disconnect x from y in Δ by removing a set $\mathcal{X}$ containing at most three points, and since $\mathcal{X}$ is finite, we have $\dim(\mathcal{X}) = 0$. From this, it follows that $\dim(\Delta) = 0 + 1 = 1$. □