Math 1100B — Calculus, Final Exam — 2009-04-06

1. Let $f(x) = \sin[\ln(x)]$ for all $x \in [1, e^{2\pi}]$, and let $F(x)$ be an antiderivative of $f(x)$. Without computing F, answer the following questions:

$(\frac{10}{200})$ (a) Find the intervals in $[1, e^{2\pi}]$ where F is increasing or decreasing.

Solution: For all $x \in \left[1, e^{2\pi}\right]$,

$$\Big(F \text{ is increasing at } x\Big) \iff \Big(F'(x) > 0\Big) \iff \Big(f(x) > 0\Big) \iff \Big(\sin[\ln(x)] > 0\Big)$$
$$\iff \Big(0 < \ln(x) < \pi\Big) \iff \boxed{\Big(1 < x < e^{\pi}\Big),}$$

Likewise,

$$\Big(F \text{ is decreasing at } x\Big) \iff \Big(\sin[\ln(x)] < 0\Big) \iff \Big(\pi < \ln(x) < 2\pi\Big) \iff \boxed{\Big(e^{\pi} < x < e^{2\pi}\Big).}$$

□

$(\frac{5}{200})$ (b) Find the local maxima and local minima of F.

Solution: F is increasing on $(1, e^{\pi})$ and decreasing on $\left(e^{\pi}, e^{2\pi}\right)$. Thus, $\boxed{x = e^{\pi} \text{ is a local maximum.}}$ The endpoints of the domain are also extremal points. F is increasing on $(1, e^{\pi})$; thus, $\boxed{x = 1 \text{ is a local minimum}}$. Likewise, F is decreasing on $\left(e^{\pi}, e^{2\pi}\right)$, so $\boxed{x = e^{2\pi} \text{ is a local minimum.}}$ □

$(\frac{10}{200})$ (c) Find the intervals where F is concave-up or concave-down.

Solution: The Chain Rule says

$$F''(x) = f'(x) = \sin'[\ln(x)] \cdot \ln'(x) = \cos[\ln(x)] \cdot \frac{1}{x} = \frac{\cos(\ln(x))}{x}.$$

Thus, for any $x \in \left[1, e^{2\pi}\right]$,

$$\Big(F \text{ is concave-up at } x\Big) \iff \Big(F''(x) > 0\Big) \iff \left(\frac{\cos(\ln(x))}{x} > 0\right) \iff \Big(\cos(\ln(x)) > 0\Big)$$
$$\iff \left(0 < \ln(x) < \frac{\pi}{2} \text{ or } \frac{3\pi}{2} < \ln(x) < 2\pi\right)$$
$$\iff \boxed{\left(1 < x < e^{\frac{\pi}{2}} \text{ or } e^{\frac{3\pi}{2}} < x < e^{2\pi}\right).}$$

Likewise,

$$\Big(F \text{ is concave-down at } x\Big) \iff \Big(\cos(\ln(x)) < 0\Big) \iff \boxed{\left(e^{\frac{\pi}{2}} < x < e^{\frac{3\pi}{2}}\right).}$$

□

$(\frac{5}{200})$ (d) Find the inflection points of F.

Solution: The inflection points are places where the concavity changes from up to down. These are at $\boxed{x = e^{\frac{\pi}{2}}}$ and $\boxed{x = e^{\frac{3\pi}{2}}.}$ □

2. Compute the following limits:

$(\frac{10}{200})$ (a) $\lim_{x\to 0} \frac{(1+x^2)(1-\cos(x)^2)}{x^2+x^4}$.

Solution: Recall that $\cos(x)^2 + \sin(x)^2 = 1$. Thus, $1 - \cos(x)^2 = \sin(x)^2$. Thus,

$$\frac{(1+x^2)(1-\cos(x)^2)}{x^2+x^4} = \frac{(1+x^2)\cdot\sin(x)^2}{x^2(1+x^2)} = \frac{\sin(x)^2}{x^2} = \left(\frac{\sin(x)}{x}\right)^2$$

Thus,

$$\lim_{x\to 0}\frac{(1+x^2)(1-\cos(x)^2)}{x^2+x^4} = \lim_{x\to 0}\left(\frac{\sin(x)}{x}\right)^2 = \left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)^2 \underset{(*)}{=} 1^2 = \boxed{1.}$$

Here, $(*)$ is because $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$ (a Lemma we proved in class, which we used to obtain the derivatives of trigonometric functions). Alternately, this fact can be easily verified through L'Hospital's rule. □

$(\frac{10}{200})$ (b) $\lim_{x\to\infty} x^{1/x}$.

Solution: This limit has indeterminate form ∞^0. The first step is to take the logarithm. We have $\ln(x^{1/x}) = \frac{1}{x}\ln(x) = \frac{\ln(x)}{x}$. Thus,

$$\lim_{x\to\infty}\ln(x^{1/x}) = \lim_{x\to\infty}\frac{\ln(x)}{x} \underset{(H)}{=} \lim_{x\to\infty}\frac{1/x}{1} = \lim_{x\to\infty}\frac{1}{x} = 0.$$

where (H) is L'Hospital's Rule. Thus,

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty}\exp[\ln(x^{1/x})] = \exp\left[\lim_{x\to\infty}\ln(x^{1/x})\right] = \exp[0] = \boxed{1.}$$

□

3. Compute the derivatives of the following functions

$(\frac{10}{200})$ (a) $f(x) = \ln(\cos(x))$.

Solution: $f'(x) = \ln'(\cos(x))\cdot\cos'(x) = \frac{1}{\cos(x)}\cdot(-\sin(x)) = \frac{-\sin(x)}{\cos(x)} = \boxed{-\tan(x).}$ □

$(\frac{10}{200})$ (b) $f(x) = \frac{e^{1/x}}{x^2}$.

Solution: $f'(x) = \frac{x^2\cdot e^{1/x}\cdot(\frac{-1}{x^2}) - e^{1/x}\cdot 2x}{(x^2)^2} = \frac{-e^{1/x} - 2xe^{1/x}}{x^4} = \boxed{\frac{-(1+2x)e^{1/x}}{x^4}}.$ □

4. Compute the following integrals.

$(\frac{15}{200})$ (a) $\displaystyle\int \frac{\sin(\ln(x))}{x}\, dx.$

Solution: $\displaystyle\int \frac{\sin(\ln(x))}{x}\, dx \underset{\text{(s)}}{=} \int \sin(u)\, du \underset{(*)}{=} -\cos(u) + C \underset{\text{(s)}}{=} \boxed{-\cos(\ln(x)) + C}.$

Here, (s) is the substitution $u := \ln(x)$, so that $du = \frac{1}{x}\, dx$. Equality $(*)$ is because $\cos'(x) = -\sin(x)$. □

$(\frac{15}{200})$ (b) $\displaystyle\int x^2 \sin(x)\, dx.$

Solution: $\displaystyle\int x^2 \sin(x)\ dx \underset{(p)}{=} -x^2\cos(x) + \int 2x\cos(x)\ dx \underset{(\P)}{=} -x^2\cos(x) + 2x\sin(x) - \int 2\sin(x)\ dx = \boxed{-x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C.}$

Here, (p) is integration by parts with $u = x^2$ and $dv = \sin(x)$, so that $du = 2x$ and $v = -\cos(x)$. Next, (¶) is integration by parts with $u = 2x$ and $dv = \cos(x)$, so that $du = 2$ and $v = \sin(x)$. □

$(\frac{15}{200})$ (c) $\displaystyle\int \sin(x)^3\cos(x)^5\ dx.$

Solution:

$$\begin{aligned}
\int \sin(x)^3\cos(x)^5\ dx &= \int \sin(x)^3 \cdot \left(\cos(x)^2\right)^2 \cdot \cos(x)\ dx \\
&\underset{(*)}{=} \int \sin(x)^3\left(1 - \sin(x)^2\right)^2 \cdot \cos(x)\ dx \underset{\text{(s)}}{=} \int u^3\left(1 - u^2\right)^2\ du \\
&= \int u^3\left(1 - 2u^2 + u^4\right)\ du = \int u^3 - 2u^5 + u^7\ du \\
&= \frac{1}{4}u^4 - \frac{1}{3}u^6 + \frac{1}{8}u^8 + C \\
&\underset{\text{(s)}}{=} \boxed{\frac{1}{4}\sin(x)^4 - \frac{1}{3}\sin(x)^6 + \frac{1}{8}\sin(x)^8 + C.}
\end{aligned}$$

Here, $(*)$ is because $\cos(x)^2 = 1 - \sin(x)^2$ because $\sin(x)^2 + \cos(x)^2 = 1$. Meanwhile, (s) is the substitution $u = \sin(x)$ so that $du = \cos(x)\ dx$.

Alternately, we could write:

$$\begin{aligned}
\int \sin(x)^3\cos(x)^5\ dx &= \int \sin(x)^2\sin(x)\cos(x)^5\ dx \underset{(*)}{=} \int (1 - \cos(x)^2)\sin(x)\cos(x)^5\ dx \\
&\underset{\text{(s)}}{=} \int -(1 - u^2)\cdot u^5\ du = \int u^7 - u^5\ du = \frac{u^8}{8} - \frac{u^6}{6} + C \\
&\underset{(*)}{=} \boxed{\frac{\cos(x)^8}{8} - \frac{\cos(x)^6}{6} + C.}
\end{aligned}$$

Here, $(*)$ is because $\cos(x)^2 = 1 - \sin(x)^2$ because $\sin(x)^2 + \cos(x)^2 = 1$. Meanwhile, (s) is the substitution $u = \cos(x)$ so that $du = -\sin(x)\ dx$. □

($\frac{15}{200}$) (d) $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$.

Solution: Let $x = 4\sin(\theta)$; then $dx = 4\cos(\theta)\,d\theta$, and

$$\begin{aligned}\sqrt{16-x^2} &= \sqrt{16-16\sin(\theta)^2} = 4\sqrt{1-\sin(\theta)^2} = 4\sqrt{\cos(\theta)^2} \\ &= 4|\cos(\theta)| \underset{(*)}{=} 4\cos(\theta),\end{aligned}$$

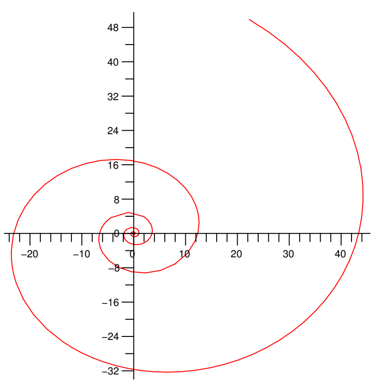
where $(*)$ assumes that $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Thus,

$$\int \frac{1}{\sqrt{16-x^2}}\,dx = \int \frac{4\cos(\theta)\,d\theta}{4\cos(\theta)} = \int d\theta = \theta = \arcsin(x/4).$$

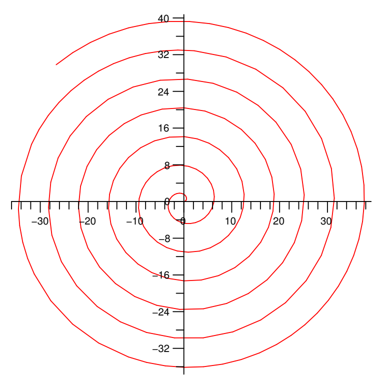
□

5. Fix a constant $b > 0$. The polar curve defined by the function $r(\theta) = e^{b\theta}$ is called a *logarithmic spiral*. Logarithmic spirals appear frequently as the trajectories of ordinary differential equations, and in the study of the complex exponential map. They also often appear in biology; for example, the shape of a nautilus shell is a logarithmic spiral.

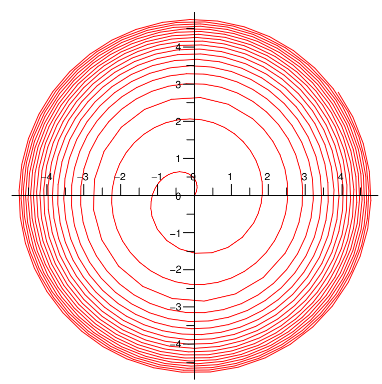
($\frac{5}{200}$) (a) *Multiple choice.* Which of the following three pictures (A,B, or C) do you think best describes the shape of the logarithmic spiral? (Do *not* use a graphing device. Instead, sketch the curve $y = e^x$ in Cartesian coordinates, and then mentally 'translate' your picture into polar coordinates).



(A)



(B)



(C)

(b) For simplicity, let $b = 1$, to get the logarithmic spiral $r(\theta) = e^\theta$. Let $x(\theta)$ and
($\frac{5}{200}$) $y(\theta)$ be the Cartesian coordinates of the point with polar coordinates (θ, e^θ). Find expressions for $x(\theta)$ and $y(\theta)$.

Solution: $x(\theta) = r(\theta)\cdot\cos(\theta) = \boxed{e^\theta\cos(\theta),}$ and $y(\theta) = r(\theta)\cdot\sin(\theta) = \boxed{e^\theta\sin(\theta).}$ □

($\frac{10}{200}$) (c) Compute the *slope* of this logarithmic spiral at the point (θ, e^θ).

Solution: We have

$$x'(\theta) = e^\theta\cos(\theta) - e^\theta\sin(\theta) = e^\theta\Big(\cos(\theta) - \sin(\theta)\Big),$$

$$\text{and } y'(\theta) = e^{\theta}\sin(\theta) + e^{\theta}\cos(\theta) = e^{\theta}\Big(\sin(\theta)+\cos(\theta)\Big).$$

$$\text{Thus, } \operatorname{slope}(\theta) = \frac{y'(\theta)}{x'(\theta)} = \frac{e^{\theta}(\sin(\theta)+\cos(\theta))}{e^{\theta}(\cos(\theta)-\sin(\theta))} = \boxed{\frac{\sin(\theta)+\cos(\theta)}{\cos(\theta)-\sin(\theta)}}.$$

□

($\frac{10}{200}$) (d) Compute the *arc length* of this logarithmic spiral from $\theta = 0$ to $\theta = 1$.

Solution: We have

$$\begin{aligned}
x'(\theta)^2 &= e^{2\theta}\Big(\cos(\theta)-\sin(\theta)\Big)^2 = e^{2\theta}\left(\cos(\theta)^2 - 2\cos(\theta)\sin(\theta)+\sin(\theta)^2\right)\\
&= e^{2\theta}\left(1-2\cos(\theta)\sin(\theta)\right)\\
\text{and } y'(\theta)^2 &= e^{2\theta}\Big(\cos(\theta)+\sin(\theta)\Big)^2 = e^{2\theta}\left(\cos(\theta)^2 + 2\cos(\theta)\sin(\theta)+\sin(\theta)^2\right)\\
&= e^{2\theta}\left(1+2\cos(\theta)\sin(\theta)\right).\\
\text{Thus, } \sqrt{x'(\theta)^2+y'(\theta)^2} &= \sqrt{e^{2\theta}\left(1-2\cos(\theta)\sin(\theta)\right)+e^{2\theta}\left(1+2\cos(\theta)\sin(\theta)\right)}\\
&= e^{\theta}\sqrt{1-2\cos(\theta)\sin(\theta)+1+2\cos(\theta)\sin(\theta)}\\
&= e^{\theta}\sqrt{2} = \sqrt{2}\,e^{\theta}.\\
\text{Thus, } \operatorname{arclength}(0,1) &= \int_0^1 \sqrt{x'(\theta)^2+y'(\theta)^2}\,d\theta = \int_0^1 \sqrt{2}\,e^{\theta}\,d\theta\\
&= \sqrt{2}\,e^{\theta}\Big|_{\theta=0}^{\theta=1} = \boxed{\sqrt{2}\,(e-1).}
\end{aligned}$$

□

6. For each of the following series, determine: is the series convergent? Is it absolutely convergent?

($\frac{10}{200}$) (a) $\displaystyle\sum_{n=1}^{\infty} \frac{\arctan(n)}{n^3}$.

Solution: For all $n \in \mathbb{N}$, we have $0 < \arctan(n) \leq \frac{\pi}{2}$. Thus,

$$0 \leq \left|\frac{\arctan(n)}{n^3}\right| = \frac{|\arctan(n)|}{n^3} \leq \frac{1}{n^3}.$$

Now, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^3}$ is convergent (it is a p-series with $p = 3 > 1$). Thus, the Comparison Test says that the series $\displaystyle\sum_{n=1}^{\infty}\left|\frac{\arctan(n)}{n^3}\right|$ converges. Thus, series $\displaystyle\sum_{n=1}^{\infty}\frac{\arctan(n)}{n^3}$ converges absolutely. □

($\frac{10}{200}$) (b) $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n\ln(n)}$. (*Hint.* Use the Integral Test).

Solution: Let $f(x) = \frac{1}{x \ln(x)}$. Then the series has the form $\sum_{n=1}^{\infty} f(n)$, so it converges if and only if the improper integral $\int_1^{\infty} f(x)\ dx$ converges. But

$$\begin{aligned}\int_1^{\infty} f(x)\ dx &= \lim_{T\to\infty} \int_1^T \frac{1}{x\ln(x)}\ dx \underset{(*)}{=} \lim_{T\to\infty} \int_e^{e^T} \frac{du}{u}\ du \\ &= \lim_{T\to\infty} \ln(x)\Big|_{x=e}^{e^T} = \lim_{T\to\infty} (T-1) = \infty.\end{aligned}$$

here $(*)$ is the substitution $x := e^u$, so that $u = \ln(x)$; hence $du = \frac{1}{x}\ dx$. Thus, the integral $\int_1^{\infty} f(x)\ dx$ diverges. Thus, the series $\sum_{n=1}^{\infty} \frac{1}{n\ln(n)}$ also diverges. □

$(\frac{10}{200})$ (c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$.

Solution: This is an alternating series of the form $\sum_{n=1}^{\infty} (-1)^n a_n$, where $a_n := \frac{1}{\sqrt{n}}$ for all $n \in \mathbb{N}$. We have $a_1 > a_2 > a_3 > a_4 > \cdots$ and $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$; thus, the series converges by the Alternating Series test. However, the series is *not* absolutely convergent, because the series

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{\sqrt{n}} \right| = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$

diverges, because it is a p-series with $p = 1/2 \leq 1$. Thus, the series is conditionally convergent. □

$(\frac{10}{200})$ 7. Consider the power series $\sum_{n=0}^{\infty} \frac{n^3 x^n}{4^n}$. Show that the *radius of convergence* of this series is $R = 4$. (*Hint:* Use the Ratio Test.)

Solution: Let $a_n := \frac{n^3 x^n}{4^n}$. Then

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^3 |x|^{n+1}/4^{n+1}}{n^3 |x|^n / 4^n} = \frac{(n+1)^3}{n^3} \frac{|x|^{n+1}}{|x|^n} \frac{4^n}{4^{n+1}} = \left(\frac{n+1}{n}\right)^2 \frac{|x|}{4} \xrightarrow[n\to\infty]{} \frac{|x|}{4}.$$

Thus,

$$\left(\sum_{n=0}^{\infty} \frac{n^3 x^n}{4^n} \text{ converges}\right) \Longrightarrow \left(\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \leq 1\right) \Longleftrightarrow \left(\frac{|x|}{4} \leq 1\right) \Longleftrightarrow \left(|x| \leq 4\right)$$

Thus, the radius of convergence is $R = 4$. □