Let $f : \mathbb{C} \to \mathbb{C}$ be the complex-inversion map, ie. $f(z) = \frac{1}{z}$.

(a) Suppose $z = x + iy$ and $f(z) = u(x, y) + v(x, y)i$ for some functions $u, v : \mathbb{R}^2 \to \mathbb{R}$. Express $u(x, y)$ and $v(x, y)$ in terms of $x$ and $y$.

Solution: Suppose $z = x + iy$. If $z = r\angle \theta$, where $r = \sqrt{x^2 + y^2}$, then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Thus,

$$f(z) = \frac{1}{r} \angle (-\theta) = \frac{1}{r} \left( \cos(-\theta) + i\sin(-\theta) \right) = \frac{1}{r^2} \cdot r \left( \cos(\theta) - i\sin(\theta) \right)$$

$$= \frac{1}{r^2} (x - iy) = \left( \frac{x}{x^2 + y^2} \right) - \left( \frac{y}{x^2 + y^2} \right) i$$

Hence, $u(x, y) = \frac{x}{x^2 + y^2}$ and $v(x, y) = \frac{-y}{x^2 + y^2}$.

(b) Show that $f$ satisfies the Cauchy-Riemann equations everywhere except at $z = 0$. Hence $f$ is analytic everywhere except at the origin.

Solution: First observe that

$$\partial_x u(x, y) = \frac{(x^2 + y^2) - x \cdot (2x)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\partial_y u(x, y) = \frac{-x \cdot (2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}.$$  

$$\partial_x v(x, y) = \frac{y \cdot (2x)}{(x^2 + y^2)^2} = \frac{2xy}{(x^2 + y^2)^2}.$$  

$$\partial_y v(x, y) = \frac{-(x^2 + y^2) + y \cdot (2y)}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$  

It follows that $\partial_x u(x, y) = \partial_y v(x, y)$ and $\partial_y u(x, y) = -\partial_x v(x, y)$, as desired.

(c) Let $z = r\angle \theta$, and let $z_1 = r_1\angle \theta$ be another nearby point with the same angle, but with $r_1 < r$.

i. Express $z - z_1$ in polar coordinates.

Then express $z_1 - z$ in polar coordinates. [Hint: Drawing a picture may help.]

Solution: $z - z_1 = (r\angle \theta) - (r_1\angle \theta) = (r - r_1)\angle \theta$. Note that $(r - r_1) > 0$ because $r_1 < r$.

Thus, $(z_1 - z) = -(z - z_1) = -1 \cdot ((r - r_1)\angle \theta) = (r - r_1)\angle (\pi \pm \theta)$.

ii. Compute $f(z)$ and $f(z_1)$.

Solution: $f(z) = \frac{1}{r} \angle (-\theta)$ and $f(z_1) = \frac{1}{r_1} \angle (-\theta)$.
iii. Express \( f(z_1) - f(z) \) in polar coordinates

**Solution:**
\[
f(z_1) - f(z) = \left( \frac{1}{r_1} \angle(-\theta) \right) - \left( \frac{1}{r} \angle(-\theta) \right) = \left( \frac{1}{r_1} - \frac{1}{r} \right) \angle(-\theta) = \left( \frac{r - r_1}{r_1 \cdot r} \right) \angle(-\theta) \]
(because \( r_1 < r \), so \( \frac{1}{r_1} > \frac{1}{r} \)).

iv. Compute \( \frac{f(z_1) - f(z)}{z_1 - z} \).

**Solution:**
\[
\frac{f(z_1) - f(z)}{z_1 - z} = \left( \frac{r - r_1}{r_1 \cdot r} \right) \angle(-\theta) \frac{(r - r_1) \angle(\theta \pm \pi)}{(r - r_1) \angle(\theta \pm \pi)} = \left( \frac{1}{r_1 \cdot r} \right) \angle(-2\theta \mp \pi).
\]

v. Take the limit as \( z_1 \to z \) (ie. as \( r_1 \to r \)) of \( \frac{f(z_1) - f(z)}{z_1 - z} \).

**Solution:**
\[
\lim_{z_1 \to z} \frac{f(z_1) - f(z)}{z_1 - z} = \lim_{r_1 \to r} \left( \frac{1}{r_1 \cdot r} \right) \angle(-2\theta \mp \pi) = \frac{1}{r^2} \angle(-2\theta \mp \pi) = -\frac{1}{z^2}.
\]

vi. Use the answer to question (v) to deduce the value of \( f'(z) \). Explain carefully why your reasoning is justified.

**Solution:** In (a) we showed that \( f \) is analytic at \( z \). Thus, \( f'(z) \) exists, and will be equal to \( \lim_{z_1 \to z} \frac{f(z_1) - f(z)}{z_1 - z} \) if \( z_1 \) approaches \( z \) from any direction. Hence, \( f'(z) = \frac{1}{z^2} \).

(d) Recall that \( f' = \partial_z u + i\partial_z v \); use this to compute \( f'(z) \), thereby confirming your answer to question (c)vi.

**Solution:**
\[
f'(z) = \partial_z u + i\partial_z v = \left( \frac{y^2 - x^2}{(x^2 + y^2)^2} \right) + i \cdot \left( \frac{2xy}{(x^2 + y^2)^2} \right) = \frac{y^2 - x^2 + 2ixy}{(x^2 + y^2)^2} = \frac{- \left( x - iy \right)^2}{(x^2 + y^2)^2} = - \left( \frac{x - iy}{x^2 + y^2} \right)^2 = - \left( \frac{1}{z} \right)^2 = -\frac{1}{z^2}.
\]

2. Let \( f, g : \mathbb{C} \to \mathbb{C} \) be analytic functions. Let \( U \subset \mathbb{C} \) be an open subset, bounded by a simple closed curve \( \Gamma \subset \mathbb{C} \). Suppose \( f(z) = g(z) \) for all \( z \in \Gamma \). Use the Maximum Modulus Principle to prove that \( f(u) = g(u) \) for all \( u \in U \).

**Solution:** Let \( h(z) = f(z) - g(z) \). Then \( h \) is an analytic function, so the Maximum Modulus Principle says that \( |h(u)| \) takes its maximal value in \( U \) on the boundary curve \( \Gamma \).

However, for all \( z \in \Gamma \), we have \( f(z) = g(z) \); hence \( h(z) = f(z) - g(z) = 0 \). Hence \( |h(z)| = 0 \) for all \( z \in \Gamma \); since this is the maximum value of \( |h(z)| \) in \( U \), we conclude that \( |h(u)| = 0 \) for all \( u \in U \). This means that \( f(u) = g(u) \) for all \( u \in U \).
3. Suppose that \( p(z) \) and \( q(z) \) are polynomials, and that \( q \) has a simple root at \( c \in \mathbb{C} \) (ie. \( q(c) = 0 \), but \( q'(c) \neq 0 \)).

Thus, if \( f(z) = \frac{p(z)}{q(z)} \), then \( f(z) \) has a simple pole at \( c \). Show that

\[
\text{Residue}(f; c) = \frac{p(c)}{q'(c)}
\]

**Solution:** Let \( g(z) = f(z) \cdot (z - c) \). Then \( g \) is analytic near \( c \), and the residue of \( f(z) \) at \( c \) is just the value of \( g(c) \). But

\[
g(c) = \lim_{z \to c} g(z) = \lim_{z \to c} (z - c) \cdot \frac{p(z)}{q(z)} = \lim_{z \to c} \frac{p(z)}{\left(\frac{q(z)}{z - c}\right)}
\]

\[
= \frac{\lim_{z \to c} p(z)}{\lim_{z \to c} \left(\frac{q(z)}{z - c}\right)} = \frac{p(c)}{q'(c)}.
\]

4. Let \( f(z) = \frac{1}{z} \). Let \( \square \subset \mathbb{C} \) be the square loop around zero with vertices at \( (1 + i), \ (1 - i), \ (-1 + i), \) and \( (-1 - i) \), as shown in Figure 1(A).

(a) Does \( f \) have any poles inside the loop \( \square \)? If so, find the residue(s) of \( f \) at this/these pole(s).

**Solution:** Yes. \( f \) has a pole at zero.

Write \( f(z) = \frac{1}{q(z)} \), where \( q(z) = z \). Then Question 3 says that

\[
\text{Residue}(f; 0) = \frac{1}{q'(0)} = \frac{1}{1} = 1.
\]
(b) Use Cauchy’s Residue Formula to compute \( \oint_C f(z) \, dz \)

**Solution:** Cauchy’s Residue Formula says that
\[
\oint_C f(z) \, dz = 2\pi i \cdot \text{Residue}(f; 0) = 2\pi i.
\]

(c) Let \( \Box = \Box \cup \mathbf{I}, \) where \( \mathbf{I} \) is the right side of the box (ie. the line from \(-1 - i\) to \(1 + i\)) and “\( \Box \)” is the other three sides of the box. [Figure 1(B)].

Use parametric integration to evaluate the path integral \( \int_{\mathbf{I}} f(z) \, dz \).

**Solution:** Parametrize \( \mathbf{I} \) by \( \gamma: [-1, 1] \rightarrow \mathbb{C} \) defined \( \gamma(t) = 1 + it \). Then
\[
\int_{\mathbf{I}} f(z) \, dz = \int_{-1}^{1} f(\gamma(t)) \gamma'(t) \, dt = \int_{-1}^{1} \frac{i}{1 + it} \, dt = i \cdot \int_{-1}^{1} \frac{1 - it}{1 + t^2} \, dt
\]
\[
= i \cdot \int_{-1}^{1} \frac{1 - it}{1 + t^2} \, dt + \frac{1}{2} \log(1 + t^2)_{-1}^{1} = i \cdot \left( \frac{n}{4} - \frac{-n}{4} \right) + \frac{1}{2} \left( \log(2) - \log(2) \right) = \frac{\pi i}{2}.
\]

(d) Assume the path integral \( \int_{\Box} f(z) \, dz \) on each of the other three sides of \( \Box \) is equal to \( \int_{\mathbf{I}} f(z) \, dz \). Use this to compute the value of \( \oint_{\Box} f(z) \, dz \).

**Solution:**
\[
\oint_{\Box} f(z) \, dz = 4 \times \int_{\mathbf{I}} f(z) \, dz = 4 \times \frac{\pi i}{2} = 2\pi i.
\]

5. Let \( n \) be an even number. Let \( F_n(z) = \frac{1}{1 + z^n} \).

(a) Let \( \omega \) be any \( n \)th root of \((-1)\). Show that the poles of \( F_n \) are the elements of the set
\[
\mathcal{P} = \left\{ 1 \angle \left( \frac{k\pi}{n} \right) ; \ k \text{ any odd number} \right\}.
\]

**Solution:** Let \( z = r \angle \theta \). Then
\[
\begin{aligned}
\text{(z is a pole of } F_n) & \iff (1 + z^n = 0) \iff (z^n = -1) \iff (r^n \angle (n\theta) = 1\angle \pi) \\
& \iff (r = 1 \text{ and } n\theta = k\pi \text{ for some odd } k) \\
& \iff (r = 1 \text{ and } \theta = k\pi/n \text{ for some odd } k).
\end{aligned}
\]
(b) Let \( \zeta = e^{\pi i/n} \) [ie. \( \zeta = 1 \angle (\pi/n) \)]. Let \( Q \) be the set of all poles of \( F_n \) in the upper half plane. Show that

\[
Q = \left\{ \zeta^{2j+1} ; 0 \leq j \leq \frac{n}{2} - 1 \right\}.
\]

**Solution:** First note that \( P = \left\{ 1 \angle \left( \frac{k\pi}{n} \right) ; k \text{ odd} \right\} = \left\{ 1 \angle \left( \frac{(2j+1)\pi}{n} \right) ; j \in \mathbb{N} \right\} \).

Now let \( p \in P \), and suppose \( p = 1 \angle \left( \frac{(2j+1)\pi}{n} \right) \). Then

\[
(p \text{ is in the upper half plane } \iff \left( \frac{(2j+1)\pi}{n} < \pi \right) \iff \left( 2j + 1 < n \right) \iff \left( j < \frac{n-1}{2} \right) \iff \left( j \leq \frac{n-2}{2} = \frac{n}{2} - 1 \right).
\]

(c) If \( p \in P \), show that \( \text{Residue} \left( F_n; p \right) = \frac{-p}{n} \) [Hint: Use question #3]

**Solution:** Write \( F_n(z) = \frac{1}{q(z)} \), where \( q(z) = 1 + z^n \). Then \( q'(z) = n \cdot z^{n-1} \), and question #3 says that

\[
\text{Residue} \left( F_n; p \right) = \frac{1}{q'(p)} = \frac{1}{n \cdot p^{n-1}}
\]

By hypothesis, \( p^n = (-1) \). Thus, \( p^{n-1} = -\frac{1}{p} \). Thus, \( \frac{1}{n \cdot p^{n-1}} = \frac{-p}{n} \).

(d) Show that \( \sum_{q \in Q} \text{Residue} \left( F_n; q \right) = \sum_{j=0}^{\frac{n}{2}-1} \frac{-\zeta^{2j+1}}{n} \)

**Solution:** \( \sum_{q \in Q} \text{Residue} \left( F_n; q \right) \stackrel{(5c)}{=} \sum_{q \in Q} \frac{-q}{n} \stackrel{(5b)}{=} \sum_{j=0}^{\frac{n}{2}-1} \frac{-\zeta^{2j+1}}{n} \)

Here, (5b) is by 5b, (5c) is by 5c.

(e) It can be shown that \( \sum_{q \in Q} \text{Residue} \left( F_n; q \right) = \frac{1}{in \sin(\frac{\pi}{n})} \). Let \( K_r = L_r + J_r \) be the curve in Figure 1(C) (assume \( r > 1 \)). Compute the path integral

\[
\oint_{K_r} F_n(z) \, dz.
\]

**Solution:** \( \oint_{K_r} F_n(z) \stackrel{(5d)}{=} 2\pi i \cdot \sum_{q \in Q} \text{Residue} \left( F_n; q \right) \stackrel{(5a)}{=} 2\pi i \cdot \left( \frac{1}{in \sin(\frac{\pi}{n})} \right) \)

Here \((*)\) is by Cauchy’s Residue Theorem, and (5d) is by by part 5d.
(f) Prove that \( \left| \int_{J_r} F_n(z) \right| < \frac{\pi r}{r^n - 1} \).

**Solution:** \[
\left| \int_{J_r} F_n(z) \right| \leq \text{length } [J_r] \cdot \max_{z \in J_r} |F_n(z)| = (\pi \cdot r) \cdot \max_{z \in J_r} \left| \frac{1}{1 + z^n} \right|.
\]

Here, \( \triangle \) is the Triangle Inequality.

(g) Now let \( r \to \infty \), and compute \( \int_{-\infty}^{\infty} \frac{1}{1 + x^n} \, dx \).

**Solution:** First note that part 5f implies that
\[
\lim_{r \to \infty} \left| \int_{J_r} F_n(z) \, dz \right| \leq \lim_{r \to \infty} \frac{\pi r}{r^n - 1} = 0. \tag{1}
\]

Also,
\[
- \frac{2\pi}{n \sin \left( \frac{\pi}{n} \right)} \overset{(\Diamond)}{=} \lim_{r \to \infty} \int_{K_r} F_n(z) \, dz = \lim_{r \to \infty} \int_{J_r} F_n(z) \, dz + \lim_{r \to \infty} \int_{L_r} F_n(z) \, dz
\]
\[
= \lim_{r \to \infty} \int_{J_r} F_n(z) \, dz + \lim_{r \to \infty} \int_{-r}^{r} \frac{1}{1 + x^n} \, dx.
\]
\[
\overset{(*)}{=} \lim_{r \to \infty} \int_{-r}^{r} \frac{1}{1 + x^n} \, dx = \int_{-\infty}^{\infty} \frac{1}{1 + x^n} \, dx.
\]

Here, \( (\Diamond) \) is by 5e, and \( (*) \) is by eqn.(1).

We conclude that \( \int_{-\infty}^{\infty} \frac{1}{1 + x^n} \, dx = \left\lfloor \frac{2\pi}{n \sin \left( \frac{\pi}{n} \right)} \right\rfloor \).

6. Figure 2 shows a ‘cross section’ of the complex plane \( \mathbb{C} \) and the Riemann sphere \( \hat{\mathbb{C}} \).

In this picture, \( z \) is a point in \( \mathbb{C} \), and \( L \) is a straight line connecting \( z \) to the north pole \( N \) of the sphere \( \hat{\mathbb{C}} \). The line \( L \) intersects the sphere \( \hat{\mathbb{C}} \) at the point \( \zeta \). If you drop a vertical line \( V \) straight down from \( \zeta \), then \( V \) intersects \( \hat{\mathbb{C}} \) again at \( \Omega \). If you draw a line from \( \hat{w} \) to \( N \), then this line intersects \( \mathbb{C} \) at \( w \).

(a) You may assume that \( \alpha \) is a right angle ie. \( \alpha = \frac{\pi}{2} \). Show that \( \gamma = \beta \).

**Solution:** Observe that the three angles of \( \triangle N0z \) are \( \pi/2 \), \( \theta \), and \( \beta \). Thus, \( \beta = \pi/2 - \theta \).

Observe that the three angles of \( \triangle NS\zeta \) are \( \alpha = \pi/2 \), \( \theta \), and \( \gamma \). Thus, \( \gamma = \pi/2 - \theta \).

Thus, \( \gamma = \beta \).

(b) Now show that \( \delta = \gamma \).

**Solution:** \( \triangle N0w \) is just the reflection of \( \triangle S0w \) across the horizontal line. Angle \( \delta \) is the image of \( \gamma \) under this reflection. Hence \( \delta = \gamma \).
(c) Conclude that the triangle \( \triangle N0w \) is similar to the triangle \( \triangle z0N \).

**Solution:** The two of the three angles of \( \triangle z0N \) are \( \pi/2 \) and \( \beta \). Thus, \( \theta = \frac{\pi}{2} - \beta \).

The two of the three angles of \( \triangle N0w \) are \( \frac{\pi}{2} \) and \( \delta = \gamma \). The third must therefore equal \( \frac{\pi}{2} - \beta = \theta \).

Thus, \( \triangle z0N \) and \( \triangle N0w \) have the same angles, so they must be similar triangles.

(d) Conclude that \(|w| = \frac{1}{|z|}|\).

**Solution:** Because \( \triangle N0w \) is similar to \( \triangle z0N \), we have

\[
\frac{|0w|}{|N0|} = \frac{|N0|}{|0z|}
\]

But \(|w| = |0w|\) and \(|z| = |0z|\), while \(|N0| = 1\) (because \( \hat{C} \) is a sphere of radius 1). Thus, we can rewrite this equation as: \(|w| = \frac{1}{|z|}\).

(e) Define \( f: \mathbb{C} \rightarrow \mathbb{C} \) by \( f(z) = 1/z \). Let \( \hat{f}: \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}} \) be the map which 'reflects' the Riemann sphere across the complex plane (eg. Figure 2, \( \hat{f}(\zeta) = \Omega \)).

For any \( z \in \mathbb{C} \), let \( \hat{z} \in \hat{\mathbb{C}} \) be its image under the stereographic projection (eg. in Figure 2, \( \hat{z} = ?? \) and \( \hat{w} = ?? \)).

Prove that \( \hat{f}(\hat{z}) = \hat{f}(\hat{w}) \).

**Solution:** In the picture, \( \zeta = \hat{z} \) and \( \Omega = \hat{w} \). In part 6d we established that \(|w| = \frac{1}{|z|}|\). Since \( \arg(w) = \arg(z) \) (they are both on the same line through the origin), it follows that \( w = 1/\bar{z} = f(z) \). Hence, we have: \( \hat{f}(\hat{z}) = \hat{w} = \Omega = \hat{f}(\zeta) = \hat{f}(\hat{z}) \).