Atlantic Electronic
Journal of Mathematics
Volume **5**, Number **1**, Winter **2012**

`http://aejm.ca`
`http://aejm.ca/rema`
pp. **1–15**

ČEBYŠEV SETS IN THE HYPERSPACE $\mathcal{K}^1$

Robert Dawson and Scotty Levy

Department of Mathematics and Computing Science
Saint Mary's University
Halifax, Nova Scotia, Canada B3H 3C3

Abstract. We characterize the Čebyšev sets in the hyperspace $\mathcal{K}^1$ of closed segments in $\mathbf{R}$ with the Hausdorff metric. These will be seen to include sets with varying dimension. We also show that an arc in $\mathcal{K}^1$ is Čebyšev if and only if it is monotone, proving the one-dimensional case of a conjecture made in [6].

1. **Introduction.** Consider the set $\mathbf{R}\times\mathbf{R}$ of all ordered pairs $\mathbf{x} = (x_1, x_2)$ of real numbers. On its own it isn't particularly interesting. Because it's a set of ordered pairs, every element has a "first coordinate" and a "second coordinate" which are real numbers, but that's all the structure that it has. To do interesting mathematics with it, we need to define some extra structure; of course, the new structure ought to respect what was already there!

One way to do this is to turn it into a vector space, usually referred to as $\mathbf{R}^2$. To do this, we create a couple of new operations. *Vector addition* is defined on any two ordered pairs as follows: $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$; the result is another ordered pair. *Scalar multiplication* starts with a real number k and an ordered pair (x_1, x_2) and the result is the ordered pair (kx_1, kx_2). For this structure to be a vector space, vector addition must be commutative:

$$\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x} \tag{1}$$

and associative:

$$(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}) \tag{2}$$

while scalar multiplication must be associative in the sense that

$$k(l\mathbf{x}) = (kl)\mathbf{x} \tag{3}$$

and distribute over both additions:

$$(k + l)\mathbf{x} = k\mathbf{x} + l\mathbf{x}\ ,\ k(\mathbf{x} + \mathbf{y}) = k\mathbf{x} + k\mathbf{y}\ . \tag{4}$$

The reader may verify that these conditions are indeed satisfied. All other properties of vector spaces – for instance, the existence and uniqueness of the zero vector – follow from these axioms.

At this point the attentive reader may ask (for instance) "what about the dot product?" or "what about convergence?" The answer is that these are not part of the structure of a "vanilla" vector space, and must be added separately. A vector space that has been given a dot product is an "inner product space"; one that has been given a concept of convergence is a "topological vector space". In familiar cases such as $\mathbf{R}^2$ there may be a single obvious way of defining such "optional extras". In other cases there may be no way, or many different ways, to perform the upgrade.

In any vector space, we can define a line segment $\overline{ab}$ to consist of all points of the form $(1 - k)a + kb$ for $k \in [0, 1]$. (If we allow k to range over all real numbers, we get the infinite line $\overleftrightarrow{ab}$.) A set in a vector space will be said to be *convex* if, whenever it contains points a and b, it contains the segment $\overline{ab}$. Convexity is a very important concept in geometry and analysis, and there are many equivalent definitions. (The curious reader is referred to *e.g.*, Benson [1] or Valentine [11].)

Another structure that we can place onto $\mathbf{R} \times \mathbf{R}$ is a "metric", a rule assigning a distance $d(x, y)$ to any pair of points. This must be strictly positive for distinct points and zero otherwise:

$$d(\mathbf{x}, \mathbf{y}) > 0,\ d(\mathbf{x}, \mathbf{x}) = 0\ . \tag{5}$$

It must be symmetric:

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x}) \tag{6}$$

and obey the triangle inequality

$$d(\mathbf{x}, \mathbf{y}) + d\mathbf{y}, \mathbf{z}) \geq d(\mathbf{x}, \mathbf{z})\ . \tag{7}$$

Some of the (infinitely many) metrics we can put onto $\mathbf{R} \times \mathbf{R}$ are:

(i) The Euclidean plane $\mathcal{E}^2$ has distance between points (x_1, x_2) and (y_1, y_2) given by $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$.

(ii) The "taxicab plane" $\mathbf{R}^2_\diamond$ has the same underlying set as the Euclidean plane but the distance is $|x_1 - y_1| + |x_2 - y_2|$, the distance that a driver in a grid of city streets, who can only go north-south or east-west, would have to drive.

(iii) The "ℓ^∞ plane" $\mathbf{R}^2_\square$ again has the same underlying set but the distance is $\max(|x_1 - y_1|, |x_2 - y_2|)$.

These metrics work nicely with the vector space structure of $\mathbf{R}^2$. Specifically, they are unaffected by translation:

$$d(\mathbf{x}, \mathbf{y}) = d(\mathbf{x} + \mathbf{z}, \mathbf{y} + \mathbf{z}) \tag{8}$$

and they scale properly:

$$d(k\mathbf{x}, k\mathbf{y}) = |k| d(\mathbf{x}, \mathbf{y})\ . \tag{9}$$

(Warning: not all metrics on the plane have these properties! The reader may enjoy discovering some that don't.) A finite-dimensional vector space with a "well-behaved" metric is called a *Minkowski space*.

All distances in a Minkowski space are determined if we know just the distances from $\mathbf{0}$, because $d(\mathbf{x}, \mathbf{y}) = d(0, \mathbf{y} - \mathbf{x})$. That is to say, the metric on such a space is given by a *norm*, $\|\mathbf{x}\| := d(0, \mathbf{x})$. In fact, we need even less data; by (9), if we know the *unit ball* $B := \{\mathbf{x}\ :\ \|\mathbf{x}\| \leq 1\}$, we can figure out the norm of any vector. Figure 1 shows the unit balls of the Euclidean, taxicab, and ℓ^∞ planes. These are respectively $\{(x_1, x_2)\ :\ x_1^2 + x_2^2 \leq 1\}$, $\{(x_1, x_2)\ :\ |x_1| + |x_2| \leq 1\}$, and $\{(x_1, x_2)\ :\ |x_1|, |x_2| \leq 1\}$.

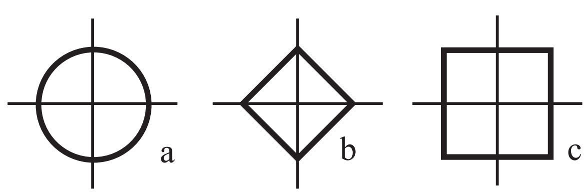

FIGURE 1. Unit balls of the Euclidean, taxicab, and ℓ^∞ planes

The unit ball of $\mathbf{R}^2_\square$ can be obtained from the unit ball of $\mathbf{R}^2_\diamond$ by rotating and scaling. This is not a coincidence; if we define

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} := \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sqrt{2} \begin{bmatrix} \sin 45^\circ & \cos 45^\circ \\ -\cos 45^\circ & \sin 45^\circ \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{10}$$

then

$$|x_1| + |x_2| = \begin{cases} |u_1| : & \text{if } x_1x_2 \geq 0 \\ |u_2| : & \text{if } x_1x_2 \leq 0 \end{cases} \tag{11}$$

$$= \max\{|u_1|, |u_2|\} . \tag{12}$$

But the matrix in (10) is $\sqrt{2}$ times a 45° rotation matrix. (Canadian readers may recall a 2008 advertising campaign for a certain breakfast cereal based on a similar observation [12].)

The unit ball is always symmetric about the origin; this is because $\|-\mathbf{x}\| = d(0, -\mathbf{x}) = |-1|d(0, \mathbf{x}) = \|\mathbf{x}\|$. It must also be convex. Let $\mathbf{x}, \mathbf{y} \in B$, and $\mathbf{z} = (1-k)\mathbf{x} + k\mathbf{y}$ (Figure 2). Then

$$\|\mathbf{z}\| \leq (1-k)\|\mathbf{x}\| + k\|\mathbf{y}\| \leq (1-k) + k = 1$$

and $\mathbf{z} \in B$ also. In the Euclidean case, B is *strictly convex*, meaning that except perhaps for its endpoints $\overline{\mathbf{xy}}$ always lies on the interior of B. In the other two cases it isn't.

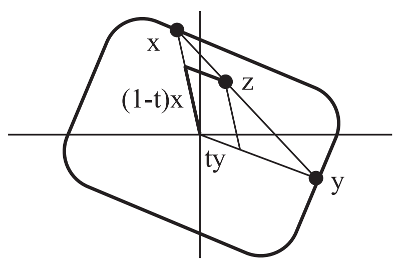

FIGURE 2. Balls are always convex

The unit ball must also (by (6)) be centrally symmetric about 0; and it must be a *body*: that is, a compact set which is the closure of its own interior. In a finite-dimensional real vector space, "compact" is the same thing as being closed and bounded. This is the famous *Heine-Borel theorem* (see any introductory analysis textbook), but readers unfamiliar with compactness may pretend, for now, that this is the definition. It may be shown (we won't) that any convex body in $\mathbf{R}^n$ that is centrally symmetric about 0 is the unit ball of some norm. Thus every such set corresponds to a Minkowski space structure on $\mathbf{R}^n$ and *vice versa*.

In a metric space, a set A will be said to "have the Čebyšev property" (or "be a Čebyšev set") if every point $\mathbf{x}$ of the space has a unique nearest neighbour $\xi_A(\mathbf{x})$ in A (the map $\xi_A(-)$ is sometimes called the "metric projection"). Intuitively, this means that every point outside the set has a smallest ball around it that touches the set, and that ball touches the set in only one point. In the Euclidean plane $\mathcal{E}^2$, the sets with this property are precisely the closed nonempty convex sets. This is interesting because the Čebyšev property is defined in terms of the metric structure of $\mathcal{E}^2$, not the vector space structure. The Euclidean plane is not unique in this regard; the observation holds in any Minkowski space with a smooth and strictly convex unit ball (see [8]). However, in other Banach spaces, neither convexity nor the Čebyšev property implies the other. Figure 3 shows that in the taxicab plane $\mathbf{R}^2_\diamond$ a Čebyšev set need not be convex , and a convex set need not be Čebyšev .

Given a metric space (X, d), we can construct a *hyperspace*. This is a collection of nonempty compact sets with a metric of its own, which would normally be based in some way on the old metric. Informally, we are treating the compact sets of the first space as "fat

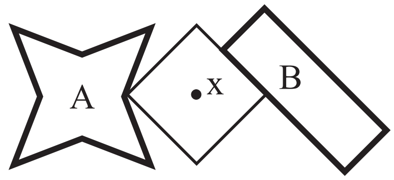

FIGURE 3. A is Čebyšev but not convex; B is convex but not Čebyšev .

points". There are many ways to do this; one that is often used is the *Hausdorff metric*. This is defined as

$$\varrho_H(A, B) := \max\{\vec{\varrho_H}(A, B), \vec{\varrho_H}(B, A)\}$$

where $\vec{\varrho_H}(X, Y)$ is the *directed Hausdorff quasimetric*

$$\vec{\varrho_H}(A, B) := \max_{x \in A} \min_{y \in B} d(x, y) \ .$$

The following may make this easier to visualize. Suppose we have a cat that lives in A and a dog that lives in B (Figure 4). The dog is curious about the cat and wants to get as close as possible; the cat wants to keep its distance. The equilibrium distance between them is the directed Hausdorff quasimetric. If also we give the cat first choice of where it lives, the distance is the Hausdorff metric.

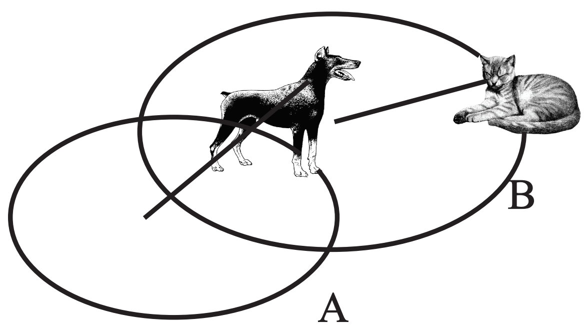

FIGURE 4. Visualizing the Hausdorff metric

Some interesting examples of hyperspaces over $\mathbf{R}^n$ are:

$\mathcal{C}^n$**::** nonempty compact sets in $\mathbf{R}^n$
$\mathcal{K}^n$**::** nonempty convex compact sets in $\mathbf{R}^n$
$\mathcal{K}_0^n$**::** convex bodies [compact sets with nonempty interior] in $\mathbf{R}^n$
$\mathcal{O}^n$**::** nonempty strictly convex compact sets in $\mathbf{R}^n$

Hyperspaces inherit a weak sort of linear structure. If A and B are compact sets we can define $kA = \{k\mathbf{x} \ : \mathbf{x} \in A\}$ (scaling by k) and $A + B = \{\mathbf{x} = \mathbf{y} : \mathbf{x} \in A, \mathbf{y} \in B\}$ (the *Minkowski sum*). These operations obey (9) in all three hyperspaces listed above, and (8) in $\mathcal{K}^n$ and $\mathcal{O}^n$. However, these are not vector space operations! In particular, $A + -A$ is only equal to $0A$ if A is a singleton. In most cases hyperspaces are infinite-dimensional, in the sense that no finite set of elements forms a basis for any neighborhood. $\mathcal{K}^1$ (which is the same as $\mathcal{O}^1$, and whose elements are just closed line segments and points) is an exception.

In general the problem of identifying the Čebyšev sets in a hyperspace seems to be very difficult. (Remember that a *set* in a hyperspace corresponds to a family of compact sets in the underlying space.) Singletons and the whole space are Čebyšev in any metric space. In $\mathcal{K}^n$ convex sets of singletons, the family $\mathcal{B}^n$ containing all balls and singletons, and any affine-convex family, are Čebyšev [3]. (Note that an affine-convex family that is not a singleton spans $\mathcal{K}^n$, and is thus infinite-dimensional for $n > 1$.) Any nested family in $\mathcal{K}^n$ such that the smaller set is always contained in the interior of the larger one is also Čebyšev [7]. In $\mathcal{O}^n$ there are more examples, such as sets of translates $\{A + t \ : \ t \in T\}$ where A is strictly convex, T convex (*ibid.*) Other Čebyšev families in $\mathcal{K}^n$ and $\mathcal{O}^n$ are given in [4, 6], but there is as yet no general characterization of them. In the remainder of this paper we will give a complete characterization of the Čebyšev sets in the hyperspace $\mathcal{K}^1$ and $\mathcal{K}^1_0$.

2. **Čebyšev sets in $\mathbf{R}^2_\diamond$ and $\mathcal{K}^1$.** We first observe that the convex compact sets in the real line admit a much simpler description than their counterparts in higher dimensions do.

Proposition 1. *The hyperspace $\mathcal{K}^1$ is isometrically equivalent to the closed upper halfplane with the taxicab metric, under the map*

$$\phi : [a_1, a_2] \mapsto \left(\frac{a_1 + a_2}{2}, \frac{a_2 - a_2}{2} \right) . \tag{13}$$

Proof: The points of $\mathcal{K}^1$ are simply closed line segments or points, and every such set is uniquely identified by its endpoints, a pair (x_1, x_2) with $x_1 \le x_2$. Conversely, every such pair defines a point (if $x_1 = x_2$) or a line segment. Thus we can identify the elements of $\mathcal{K}^1$ with the points of a closed half-plane, bounded by the line $x_1 = x_2$.

We now need to find the Hausdorff distance from $[a, b]$ to $[c, d]$. To do this, we first find the directed distance

$$\vec{\varrho_H}([a, b], [c, d]) = \max_{x\in[a,b]} \min_{y\in[c,d]} |x - y| \ .$$

If $c \le a$ and $b \le d$, this is 0; otherwise it is the larger of $c - a$ and $b - d$. Equivalently,

$$\vec{\varrho_H}([a, b], [c, d]) = \max\{(c - a), (b - d), 0\} \ .$$

By the same argument, the other directed distance is given by

$$\vec{\varrho_H}([c, d], [a, b]) = \max\{(a - c), (d - b), 0\} \ .$$

Combining these:

$$\begin{aligned} \varrho_H([a, b], [c, d]) &= \max\{\vec{\varrho_H}([a, b], [c, d]), \vec{\varrho_H}([c, d], [a, b])\} \\ &= \max\{(c - a), (a - c), (b - d), (d - b), 0\} \\ &= \max\{|a - c|, |b - d|\} \end{aligned}$$

and this is just the ℓ^∞ distance between (a, b) and (c, d).

Thus, $\mathcal{K}^1$ can be represented - including its metric structure - as $\{(x_1, x_2) : x_1 \le x_2\}$ in $\mathbf{R}^2_\square$; or, equivalently, as the closed upper half $\overline{U}$ of the taxicab plane $\mathbf{R}^2_\diamond$ (Figure 5.) In the latter representation, x_1 is the midpoint of the line segment, x_2 its radius (that is, half its length.) ■

Corollary 1. *The hyperspace K^1_0 is isometrically equivalent, under a restriction of ϕ, to the open upper halfplane U with the taxicab metric.*

Proof: The hyperspace $\mathcal{K}^1_0$ differs from $\mathcal{K}^1$ only by the omission of singletons; it thus corresponds to the points $(x_1, x_2) \in \mathbf{R}^2_\square$ with $x_1 < x_2$, or to the points $(x_1, x_2) \in \mathbf{R}^2_\diamond$ with $x_2 > 0$. ■

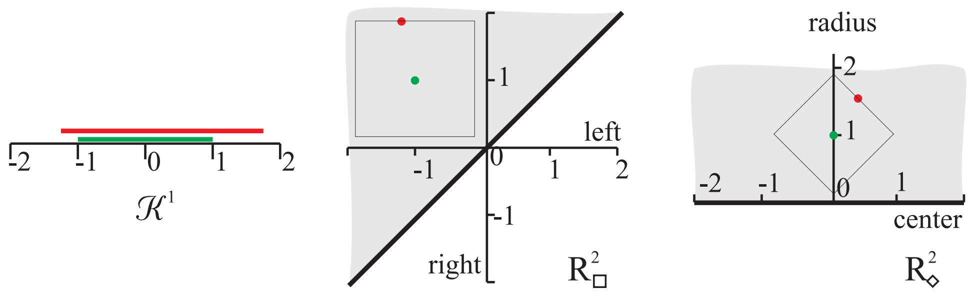

FIGURE 5. Three equivalent representations of $\mathcal{K}^1$.

We now ask, exactly what are the Čebyšev sets in $\mathbf{R}^2_\diamond$, and in the closed upper half-planes? As shown in Figure 3, they are not the same as the convex sets; but a complete characterization, while not difficult, does not appear to be in the literature. Clearly such a set S must be closed, as if it has a boundary point that is not an element, that point has no closest neighbour.

Lemma 1. *Let S be a Čebyšev set in $\mathbf{R}^2_\diamond$ or $\overline{U}$, and let $\mathbf{s} = \xi_S(\mathbf{x})$, with $r = d(\mathbf{s}, \mathbf{x})$. Then no other point of S lies in the closed cone $\bigcup_{y \in B_r(\mathbf{x})} \overrightarrow{\mathbf{sy}}$ generated at $\mathbf{s}$ by the ball $B_r(\mathbf{x})$.*

Proof: The point $\mathbf{s}$ may be a vertex of the ball (Figure 6a) or on the relative interior of an edge (Figure 6b); the cone (in $\mathbf{R}^2_\diamond$) is respectively a quadrant or a halfplane. In either case it is the union of a nested family of balls with $\mathbf{s}$ on their boundaries. If this has another intersection with S, which is closed, there is a smallest ball for which this happens; and the center of that ball has two or more closest neighbours in S.

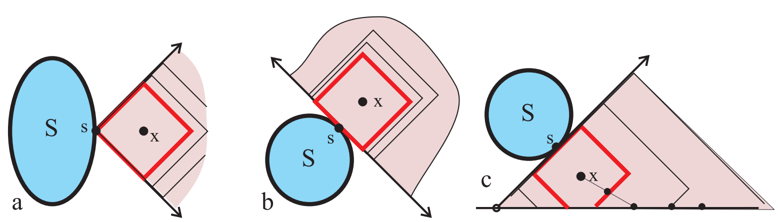

FIGURE 6. Cones that cannot contain another point of S

In $\overline{U}$ the same argument holds, except that we must restrict the centers of the family of balls to lie within the closed halfplane (Figure 6c). ■

Proposition 2. *If S is a Čebyšev set in $\mathbf{R}^2_\diamond$ or $\overline{U}$, then for each of the four linear functions $f_{\pm\pm} : \mathbf{x} \mapsto \pm x_1 \pm x_2$, there is at most one point $\mathbf{p}_{\pm\pm}$ maximizing $f_{\pm\pm}$ over S.*

Suppose that $\mathbf{p}_{++}$ is a local maximum of the function $\mathbf{x} \mapsto x_1 + x_2$ for $\mathbf{x} \in B_\epsilon(\mathbf{p}_{++}) \cap S$. Let $\mathbf{x} = \mathbf{p}_{++} + (\epsilon, \epsilon)$ for some small positive ϵ; then $\mathbf{p}_{++} = \xi_S(\mathbf{x})$, and by Lemma 1 $\mathbf{p}_{++}$ is a unique global maximum (Figure 7a). Virtually the same argument holds for the other three choices of sign: if $f_{\pm\pm}$ has a local maximum on S, there is a unique global maximum $\mathbf{p}_{\pm\pm}$. The only variation necessary occurs for $\mathbf{p}_{\pm-}$ in $\overline{U}$ if $p_2 = 0$ (Figure 7b). In this case we must take $\mathbf{x} = \mathbf{p}_{\pm-} + (\pm\epsilon, 0)$. ■

We will call $\mathcal{P}_S := \{\mathbf{p}_{\pm\pm} : \mathbf{p}_{\pm\pm} \text{ exists in } S\}$ the set of *cardinal points* of S. In $\mathbf{R}^2_\diamond$, it may be verified that all 2^4 subsets of $\{\mathbf{p}_{\pm\pm}\}$ can occur. Moreover, it is possible that a pair

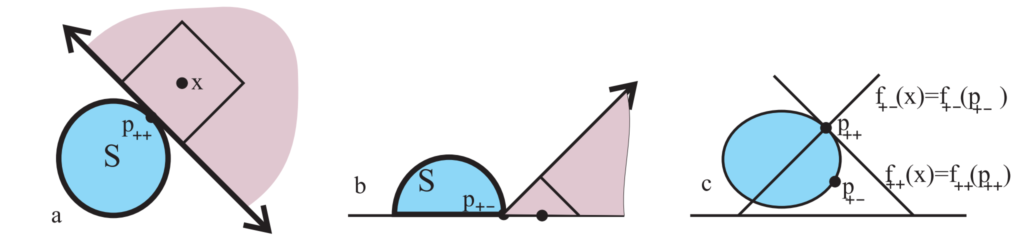

FIGURE 7. Unique corner points

of cardinal points will coincide; *e.g.*, we may have $\mathbf{p}_{++} = \mathbf{p}_{+-}$. (If two "opposite" points such as $\mathbf{p}_{++}$ and $\mathbf{p}_{--}$ coincide, then f_{--} is maximized – and f_{++} minimized – at the same unique point at which f_{++} is maximized, and S is a singleton.)

In $\overline{U}$, the second coordinate is necessarily bounded below; thus if (for instance) $\mathbf{p}_{++}$ exists and the function f_{++} is bounded above on S, then f_{+-} is also. Moreover, the set $\{\mathbf{x} \in S : f_{+-}(\mathbf{x}) > f_{+-}(\mathbf{p}_{++})\}$ is nonempty and compact (Figure 7c); thus $\mathbf{p}_{+-}$ exists. In general, we have:

Proposition 3. *For any Čebyšev set in $\overline{U}$, if $\mathbf{p}_{\pm+}$ exists, so does $\mathbf{p}_{\pm-}$.*

This leaves 9 legal subsets of $\{\mathbf{p}_{\pm\pm}\}$ that may exist; it may be verified that all are possible. Again, pairs may coincide.

We now give a sufficient condition for $\xi_S(\mathbf{x})$ to be in $\mathcal{P}_S$.

Proposition 4. *If S is Čebyšev in $\mathbf{R}^2_\diamond$ or $\overline{U}$, and if both coordinates of $\xi_S(\mathbf{x})$ differ from the corresponding coordinates of $\mathbf{x}$, then $\xi_S(\mathbf{x}) \in \mathcal{P}_S$. Specifically, if $(\xi_S(\mathbf{x}))_1 < x_1$, $(\xi_S(\mathbf{x}))_2 < x_2$, then $\xi_S(\mathbf{x}) = \mathbf{p}_{++}$ (etc).*

Proof: Suppose that $(\xi_S(\mathbf{x}))_1 < x_1$, $(\xi_S(\mathbf{x}))_2 < x_2$. Then $\xi_S(\mathbf{x})$ lies on the relative interior of the lower left edge of a ball about $\mathbf{x}$ which contains no other points of S (as in Figure 6b.) Thus $\xi_S(\mathbf{x})$ is thus a local maximum for f_{++}, and hence by 1 must be $\mathbf{p}_{++}$. ■

Stating this contrapositively, if $\mathbf{s} := \xi_S(\mathbf{x}) \notin \mathcal{P}_S$, then $\mathbf{s}$ has a coordinate in common with $\mathbf{x}$. This motivates the following definitions: if $x_1 = y_1$ and $x_2 < y_2$, we write $\mathbf{x} < \mathbf{y}$, and if $x_1 < y_1$ and $x_2 = y_2$ we write $\mathbf{x} \prec \mathbf{y}$.

Proposition 5. *If S is Čebyšev in $\mathbf{R}^2_\diamond$ or $\overline{U}$, suppose $\mathbf{x}$ and $\xi_S(\mathbf{x})$ have a coordinate in common: without loss of generality, $\mathbf{x} > \xi_S(\mathbf{x})$. Then every point of the ray $\overrightarrow{\xi_S(\mathbf{x}), \mathbf{x}}$ is mapped to $\xi_S(\mathbf{x})$ by ξ_S. In particular, the ray contains no other point of S.*

Proof: This follows from Proposition 4 if $\mathbf{s} \in \mathcal{P}_S$; otherwise it follows from the "vertex case" of Lemma 1 (Figure 6a). The point $\mathbf{x}$ is on the central ray of a right-angled cone that contains no other points of S; thus $\mathbf{s}$ is the closest point. ■

Proposition 6. *Let S be Čebyšev in $\mathbf{R}^2_\diamond$ or $\overline{U}$. Then, if $x_1 < z_1$, $\xi_S(\mathbf{x}) < \mathbf{x}$, and $\xi_S(\mathbf{z}) < \mathbf{z}$, for any $y_1 \in (x_1, z_1)$ there exists $\mathbf{y} = (y_1, y_2)$ with $\xi_S(\mathbf{y}) < \mathbf{y}$.*

Proof: Let $y_2 = \max(x_2, z_2) + (z_1 - x_1)$. First we note that $y_1 > x_1 > (p_{-+})_1$, so $\xi_S(\mathbf{y}) \neq \mathbf{p}_{-+}$; the same argument rules out $\mathbf{p}_{++}$. Again, $f_{--}(y) < f_{--}(x) < f_{--}(\mathbf{p}_{--})$, so that point and $\mathbf{p}_{+-}$ are eliminated. Suppose $\xi_S(\mathbf{y}) \prec \mathbf{y}$; then the ray $\overrightarrow{\xi_S(\mathbf{y})\mathbf{y}}$ intersects the ray $\overrightarrow{\xi_S(\mathbf{z})\mathbf{z}}$, which by Proposition 5 is impossible (Figure 8). Finally, suppose that

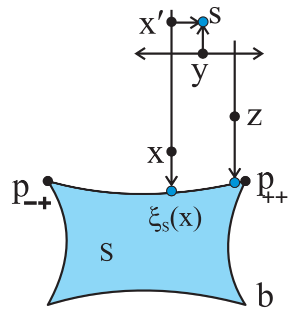

FIGURE 8. Locations of nearest points

$\mathbf{y} < \mathbf{s} = \xi_S(\mathbf{y})$; there is a point $\mathbf{x}' := (x_1, (\xi_S(\mathbf{y})_2)$ which is in the ray $\overrightarrow{\xi_S(\mathbf{x})\mathbf{x}}$ but closer to $\xi_S(\mathbf{y})$ than to $\xi_S(\mathbf{x})$, contradicting Proposition 5.

This argument carries over unchanged for all other cases in $\mathbf{R}^2_\diamond$. In $\overline{U}$, if $\mathbf{x} < \xi_S(\mathbf{x})$ and $\mathbf{z} < \xi_S(\mathbf{z})$, take $y_2 = 0$, which avoids the possibility that $\xi_S(\mathbf{y})$ might be below $\mathbf{y}$. ■

Combining this with Proposition 5 we see that the set $\{\mathbf{x} : \xi_S(\mathbf{x}) < \mathbf{x}\}$ is, if nonempty, unbounded above; and to the left and right it is either unbounded or bounded by vertical rays. It is clearly bounded below.

Proposition 7. *If S is Čebyšev in $\mathbf{R}^2_\diamond$ or $\overline{U}$, $\xi_S(\mathbf{x}) < \mathbf{x}$ and $\xi_S(\mathbf{y}) < \mathbf{y}$, then*

$$|(\xi_S(\mathbf{x}))_2 - (\xi_S(\mathbf{y}))_2| < |(\xi_S(\mathbf{x}))_1 - (\xi_S(\mathbf{y}))_1| \tag{14}$$

and

$$||\xi_S(\mathbf{x}) - \xi_S(\mathbf{y})|| < 2||\mathbf{x} - \mathbf{y}|| \; . \tag{15}$$

Proof: The inequality (14) follows immediately from Lemma 1, and

$$\begin{aligned} ||\xi_S(\mathbf{x}) - \xi_S(\mathbf{y})|| &= |(\xi_S(\mathbf{x}))_2 - (\xi_S(\mathbf{y}))_2| + |(\xi_S(\mathbf{x}))_1 - (\xi_S(\mathbf{y}))_1| \\ &< 2|(\xi_S(\mathbf{x}))_1 - (\xi_S(\mathbf{y}))_1| \qquad \text{by (14)} \\ &= 2|x_1 - y_1| \\ &\leq 2||\mathbf{x} - \mathbf{y}|| \end{aligned}$$

proving (15). ■

Corollary 2. *The lower boundary of $\{\mathbf{x} : \xi_S(\mathbf{x}) < \mathbf{x}\}$ (that is, the set $\{\xi_S(\mathbf{x}) : \xi_S(\mathbf{x}) < \mathbf{x}\}$) is the graph $\{\mathbf{x} : x_2 = f(x_1)\}$ of a distance-decreasing function $I \to \mathbf{R}$ where I is a closed interval, closed ray, or $\mathbf{R}$.*

We are now in possession of a set of conditions that are not only necessary but sufficient for a set in $\mathbf{R}^2_\diamond$ or $\overline{U}$ to be Čebyšev .

Theorem 1. *A nonempty closed set S in $\mathbf{R}^2_\diamond$ or $\overline{U}$ is Čebyšev if and only if its boundary is the union of a subset, possibly empty, of the following:*

- *a curve S_t of the form $\{\mathbf{x} : x_1 \in I_+, x_2 = f_+(x_1)\}$, where I_+ is a point, a segment, a closed ray, or $\mathbf{R}$; f_+ is distance-decreasing; and if $s \in S$, $s_1 = x_1$, then $s_2 \leq x_2$;*
- *a curve S_r of the form $\{\mathbf{x} : x_2 \in J_+, x_1 = g_+(x_2)\}$, where J_+ is a point, a segment, a closed ray, or $\mathbf{R}$; g_+ is distance-decreasing; and if $s \in S$, $s_2 = x_2$, then $s_1 \leq x_1$;*

- *a curve* S_b *of the form* $\{\mathbf{x} : x_1 \in I_-, x_2 = f_-(x_1)\}$*, where* I_- *is a point, a segment, a closed ray, or* $\mathbf{R}$*;* f_- *is distance-decreasing; and if* $s \in S$*,* $s_1 = x_1$*, then* $s_2 \geq x_2$*;*
- *a curve* S_l *of the form* $\{\mathbf{x} : x_2 \in J_-, x_1 = g_-(x_2)\}$*, where* J_- *is a point, a segment, a closed ray, or* $\mathbf{R}$*;* g_- *is distance-decreasing; and if* $s \in S$*,* $s_2 = x_2$*, then* $s_1 \geq x_1$*.*

Proof: We have shown above that any Čebyšev set has such a boundary. If none of the four bounding curves exists, S is the entire space and trivially Čebyšev. Otherwise, suppose, without loss of generality, that S_t exists. If $I_+ = [a, b]$, extend f_+ to all of $\mathbf{R}$ by letting $f(t) = f(a)$ for $t < a$ and $f(t) = f(b)$ for $t > b$; if $I_+ = (-\infty, b]$ or $[a, \infty)$ extend it in one direction only. Call the graph of this function $\overline{S_t}$. Do the same for any other bounding curves that may exist.

For every point $\mathbf{s} \in S$ there is a unique $\mathbf{x} \in \overline{S_t}$ with $x_1 = s_1$; then $x_2 \geq s_2$. We deduce that S_t is everywhere above or coincident with S_b; and in particular that no point can be strictly on the opposite side of both curves from S. We also note that if *e.g.* $\overline{S_t}$ and $\overline{S_r}$ meet, they can do so at only one point $\mathbf{p}$, which must be in S. Moreover, the line $\{\mathbf{x} : f_{++}(\mathbf{x}) = f_{++}(\mathbf{p})\}$ cannot meet any other point of S, so $\mathbf{p} = \mathbf{p}_{++}$.

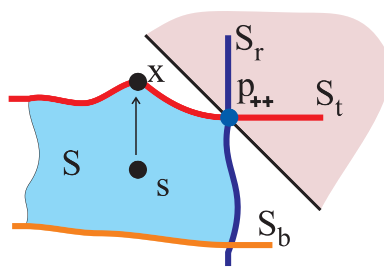

FIGURE 9. Positions of the extended boundaries

Now, as the union of the four curves contains bdS, any point in S^c is strictly on the opposite side of at least one of these curves from S. As observed above, no point is strictly on the opposite side of both $\overline{S_l}$ and $\overline{S_r}$, or of $\overline{S_t}$ and $\overline{S_b}$. Thus a point can be on the opposite side of at most two curves, and they must be $\overline{S_t}$ and $\overline{S_r}$, $\overline{S_r}$ and $\overline{S_b}$, $\overline{S_b}$ and $\overline{S_l}$, or $\overline{S_l}$ and $\overline{S_t}$.

If *e.g.* a point (x_1, x_2) is on the far side of both $\overline{S_t}$ and $\overline{S_r}$ from S, its nearest point in S is $\mathbf{p}_{++}$,and this is unique by Proposition 2. If it is on the far side of only (say) $\overline{S_t}$, then its nearest point in S is $(x_1, f_+(x_1))$ and this is unique by Lemma 1. ■

Proposition 8. *Let* S *be Čebyšev in* $\mathbf{R}^2_\diamond$ *or* $\overline{U}$*. If* S_t *is unbounded to the right and* S_r *exists, they have slant asymptotes* $x_1 = x_2 + b_t$ *and* $x_1 = x_2 + b_r$ *respectively, with* $b_t \geq b_r$*.*

Proof: If S_t is unbounded to the right, there is no point $\mathbf{p}_{++}$; thus if S_r exists, it is unbounded above. The function $x \mapsto f_+(x)$ is thus defined for all large enough x. As f_+ is distance decreasing, the function $x \mapsto (f_+(x) - f_+(0)) - (x - 0)$ is strictly decreasing. Thus either $\lim_{x\to\infty} f_+(x) - x = k$ for some finite k, or $\lim_{x\to\infty} f_+(x) - x = -\infty$. In the first case there is a 45° slant asymptote; in the second case the curve contains points for which $x_2 - x_1 < k$ for all k. By the same argument S_r either has a 45° slant asymptote or contains points for which $x_2 - x_1$ is arbitrarily large. Unless they both have slant asymptotes as described above (see Figure 10a below) they must meet, which is impossible. ■

Corollary 3. *In* $\overline{U}$*, suppose* S_b *is unbounded to the right; then* S_r *does not exist, and if* S_t *exists it is unbounded to the right.*

Proof: By Proposition 8, if S_b were unbounded to the right and S_r existed, they would have slant asymptotes approaching $(\infty, -\infty)$, which is impossible in $\overline{U}$. ■

Example 1. The set $\{\mathbf{x} : |x_1^2 - x_2^2| \le 1\}$ is Čebyšev in $\mathbf{R}^2_\diamond$; all four boundary curves exist and are doubly infinite in extent (Figure 10a).

Example 2. The set $\{\mathbf{x} : x_2^2 - x_1^2 \le 1\}$ is Čebyšev in $\overline{U}$(Figure 10b).

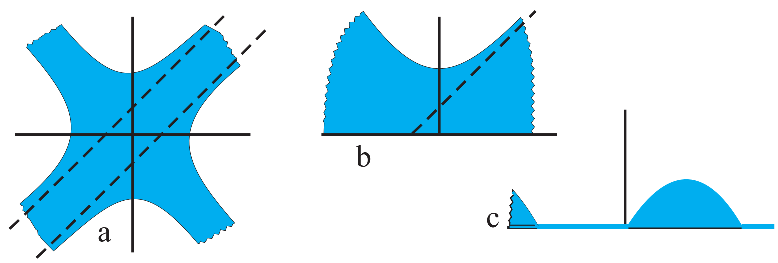

FIGURE 10. Examples of Čebyšev sets in $\mathbf{R}^2_\diamond$ and $\overline{U}$

Note that it is possible that (for instance) S_b and S_t intersect, even in their relative interiors.

Example 3. The set $\{\mathbf{x} : x_2 \le \max(\sin(x_1), 0)\}$ is Čebyšev in $\overline{U}$ (Figure 10c).

Observation 1. This set is locally one-dimensional in some neighborhoods and locally two-dimensional elsewhere. The first author has conjectured in lectures (and implicitly in [5]) that a Čebyšev set in $\mathcal{K}^n$ must have a invariant local dimension; Example 3 provides a counterexample.

Proposition 9. *Let S be Čebyšev in $\overline{U}$; unless S is a singleton, at most one of $S_b \cap S_t$ and $S_l \cap S_r$ can be nonempty.*

Proof: If $\mathbf{s} \in S_b$ then, by Lemma 1, for any other $\mathbf{x} \in S$ we have $|x_1 - s_1| > x_2 - s_2$. Thus if $\mathbf{s} \in S_b \cap S_t$, then

$$|x_1 - s_1| > |x_2 - s_2| \ . \tag{16}$$

But by the same argument, if $\mathbf{s} \in S_l \cap S_r$, then $|x_1 - s_1| < |x_2 - s_2|$ and this and (16) cannot be simultaneously satisfied. ■

An *arc* is a continuous $1-1$ image of an (open or closed) interval of the real line. Special Čebyšev arcs in hyperspaces were considered in [3, 7, 4]. In [6] the idea of a *monotone arc* in $\mathcal{K}^n$ was introduced; an arc $A = (A_t : t \in I)$ is defined to be monotone if, for every nonzero vector u, $\max_{a \in A_t} u \cdot a$ is a monotone function of t (possibly constant;) and if constant, the points a_t at which the maximum is achieved must be different for each t. It was shown in [6] that for such an arc, when one of the functions $\max_{a \in A_t} u \cdot a$ is nonconstant, that function is strictly monotone. In $\mathcal{K}^1$ the situation is even simpler; an arc is a parametrized set of intervals $[l(t), r(t)]$, and it is monotone if and only if $l(-)$ and $r(-)$ are strictly monotone functions. If they are strictly monotone in opposite senses the arc is *strongly nested*; otherwise we will call it *antinested*.

It was shown in [6] that in $\mathcal{O}^n$ all monotone arcs have the Čebyšev property, while in $\mathcal{K}^n$, $n > 1$, those (and only those) that are nested or consist of singleton sets do. The hyperspace $\mathcal{K}^1$ is an exception because its elements are all strictly convex; so (*e.g.*) the translates of a given interval form a Čebyšev arc. It was conjectured in [4] that all Čebyšev arcs in $\mathcal{K}^n$ or $\mathcal{O}^n$ are monotone; we now prove this conjecture for $\mathcal{K}^1$.

Theorem 2. *An arc in $\mathcal{K}^1$ is Čebyšev if and only if it is monotone.*

Proof: As observed above, it suffices to show that a Čebyšev arc is monotone. An arc $(\mathbf{a}_t : t \in I)$ in $\mathbf{R}^2_\diamond$ or $\overline{U}$ is Čebyšev if and only if it has no secant line with a slope of $\pm 45°$; that is,

$$|(a_s)_1 - (a_t)_1| \neq |(a_s)_2 - (a_t)_2| \,. \tag{17}$$

For if there is such a secant, then its intersection with the arc is closed; there thus exists either a closed interval in which the arc coincides with the secant, or an open interval in which the arc lies entirely on one side of the secant. In either case (Figure 11) we can construct a point with multiple nearest neighbours in the arc. Conversely, if there is no such secant, the Čebyšev property follows immediately from Lemma 1.

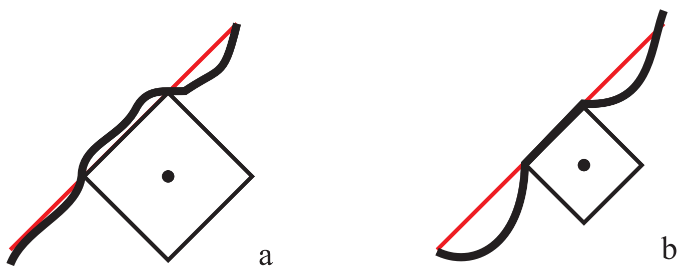

FIGURE 11. Arcs with a 45° secant are not Čebyšev

Under the isometric equivalence map $\overline{U} \to \mathcal{K}^1$, the point $\mathbf{a}_t$ maps to an interval with endpoints $l(t) = (a_t)_1 - (a_t)_2$ and $r(t) = (a_t)_1 + (a_t)_2$. If (17) holds, then for all distinct $s, t \in I$,

$$\begin{aligned} l(s) - l(t) &= ((a_s)_1 - (a_s)_2) - ((a_t)_1 - (a_t)_2) \\ &= ((a_s)_1 - (a_t)_1) - ((a_s)_2 - (a_t)_2) \\ &\neq 0 \end{aligned} \tag{18}$$

so $l(-)$ is 1-1; and

$$\begin{aligned} r(s) - r(t) &= ((a_s)_1 + (a_s)_2) - ((a_t)_1 + (a_t)_2) \\ &= ((a_s)_1 - (a_t)_1) + ((a_s)_2 - (a_t)_2) \\ &\neq 0 \end{aligned} \tag{19}$$

giving the same result for $r(-)$. Conversely, if both $l(-)$ and $r(-)$ are 1-1, then (18) and (19) are both nonzero, yielding (17). Finally, we note that a continuous function on an interval is strictly monotone if and only if it is 1-1, and we are done. ■

Translating Theorem 1 into terms appropriate to $\mathcal{K}^1$, we obtain:

Theorem 3. *A set $S \in \mathcal{K}^1$ is Čebyšev if and only if its boundary is the union of a subset, possibly empty, of:*

- *an antinested arc S_t such that if $[x,y] \in S$, $[p,q] \in S_t$, and $x + y = p + q$, then $p \leq x \leq y \leq q$;*
- *an antinested arc S_b such that if $[x,y] \in S$, $[p,q] \in S_b$, and $x + y = p + q$, then $x \leq p \leq q \leq y$;*
- *a strongly nested arc S_l such that if $[x,y] \in S$, $[p,q] \in S_l$, and $y - x = q - p$,then $p \leq x \leq q \leq y$; and*

- *a strongly nested arc S_r such that if $[x,y] \in S$, $[p,q] \in S_r$, and $y-x=q-p$,then $x \leq p \leq y \leq q$.*

We now consider the hyperspace $\mathcal{K}^1_0$ of proper intervals, which is isometrically equivalent to the open upper halfplane U. While the extension of the results above to U and $\mathcal{K}^1_0$ is simple, the proof is somewhat technical.

Theorem 4. *A relatively closed set $S \in U$ is Čebyšev if and only if its closure $\overline{S}$ in $\overline{U}$ is Čebyšev and $\overline{S}_t \subset U$.*

Proof: Suppose $\overline{S}$ is a Čebyšev set. Then for any $\mathbf{x} \in U$, either $x_2 \leq (\xi_{\overline{S}}(\mathbf{x}))_2$, in which case $\xi_{\overline{S}}(\mathbf{x}) \in U$; or $x_2 > (\xi_{\overline{S}}(\mathbf{x}))_2$, in which case $\xi_{\overline{S}}(\mathbf{x}) \in \overline{S}_t \subset U$. In either case $\xi_{\overline{S}}(\mathbf{x}) \in S$, and clearly no other point of S is as close.

Conversely, suppose S to be Čebyšev in U. We first note that $d(\mathbf{x},\overline{S}) = d(\mathbf{x},S)$; and as $\overline{S}$ is locally compact, this distance (which we shall call $d_{\min}$) is achieved. It remains to be shown that it is achieved exactly once. Let B be the horizontal axis bounding U; and let S^B be $\overline{S} \cap B$.

For $\mathbf{x} \in U$, by hypothesis $d_{\min}$ is attained exactly once in S. Suppose $||\mathbf{s}-\mathbf{x}|| = d_{\min}$ for some $s \in S^B$. Then (Figure 12a) there exists $\mathbf{x}' \in U$ that has $\mathbf{s}$ as its unique nearest neighbour in $\overline{S}$; but then $d(\mathbf{x}',S)$ is not attained, a contradiction. We conclude that for such points the distance $d_{\min}$ is attained exactly once in $\overline{S}$.

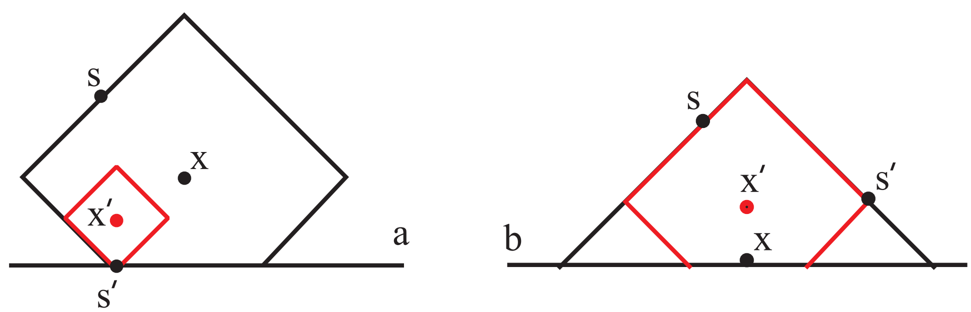

FIGURE 12. Čebyšev sets in the open upper half plane U

Consider now $\mathbf{x} = (x_1, 0) \in B$. Suppose that two points $\mathbf{s},\mathbf{s}' \in S$ are at distance $d_{\min}$ from $\mathbf{x}$. Then $|s_1 - x_1| + s_2 = |s'_1 - x_1| + s'_2 = d_{\min}$; and they are also equidistant from $\mathbf{x}' := (x_1, \min\{s_2, s'_s\})$ Figure 12b). But as $\mathbf{x}' \in U$, this contradicts our hypothesis that S is Čebyšev in U.

Suppose $||\mathbf{s}-\mathbf{x}|| = d_{\min}$ for $\mathbf{s} \in S^B$; without loss of generality, $\mathbf{s} = (s_1, 0)$ where $s_1 = x_1 - d_{\min}$. Then $\mathbf{s}$ is approached by points $\mathbf{s}_i \in S$ with $|(s_i)_1 - s_1| \leq (s_i)_2$. If not, for some $\epsilon > 0$ there exists an ϵ neighbourhod of $\mathbf{s}$ which is free of such points (Figure 13a). Thus, if we let $\mathbf{x}' := (s_1 + \epsilon/2, \epsilon/2)$, we have $d(\mathbf{x}', S) = d(\mathbf{x}', \overline{S}) = d(\mathbf{x}', \mathbf{s}) = \epsilon$ and no other point of $\overline{S}$ achieves this. But this distance is not achieved by any point of S, again contradicting our hypothesis that S is Čebyšev.

Consider the point $\mathbf{a} := (x_1, d_{\min})$. It is not in $\overline{S}$, which is closed; thus for any small enough $\epsilon > 0$ there exists an ϵ-ball about $\mathbf{a}$ disjoint from $\overline{S}$ (Figure 13b). For the same ϵ, choose some $\mathbf{s}_i$ from the sequence exhibited above, with $||\mathbf{s}_i - \mathbf{s}|| < \epsilon/2$. For $t \in \mathbf{R}$, let $\mathbf{y}(t) := (x_1 - t, \epsilon/2)$. Then

$$\overline{B}_{d-\epsilon/2}(\mathbf{y}(0)) \subset \overline{B}_d(\mathbf{x})$$

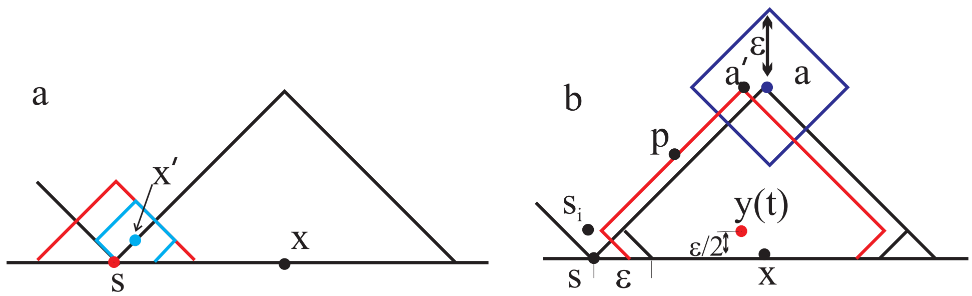

FIGURE 13. Čebyšev sets in the open upper half plane U

and

$$\overline{B}_{d-\epsilon/2}(\mathbf{y}(\epsilon/2)) \cap S \neq \emptyset\ .$$

Let $t_0 < \epsilon/2$ be the least t for which

$$\overline{B}_{d-\epsilon/2}(\mathbf{y}(t)) \cap S \neq \emptyset\ .$$

This intersection (by hypothesis, a singleton) does not come within ϵ of $\mathbf{a}$, hence does not include the apex $\mathbf{a}' := (x_1 - t_0, d)$ of $\overline{B}_{d-\epsilon/2}(\mathbf{y}(t))$. Proceeding as in Lemma 1, we can construct a continuous nested family of balls with radius $d-\epsilon/2+t$ and center $(x_1-t_0+t, \epsilon/2)$ (Figure 14a). Note that their union cannot contain any other points of S, for otherwise the center of the smallest ball to do so would have two nearest neighbours in S. But this union covers every point of $\overline{B}_d(\mathbf{x}) \setminus \overline{B}_\epsilon(\mathbf{s})$; as we can choose ϵ to be arbitrarily small, we conclude that $\overline{B}_d(\mathbf{x}) \cap \overline{S} = \{\mathbf{s}\}$.

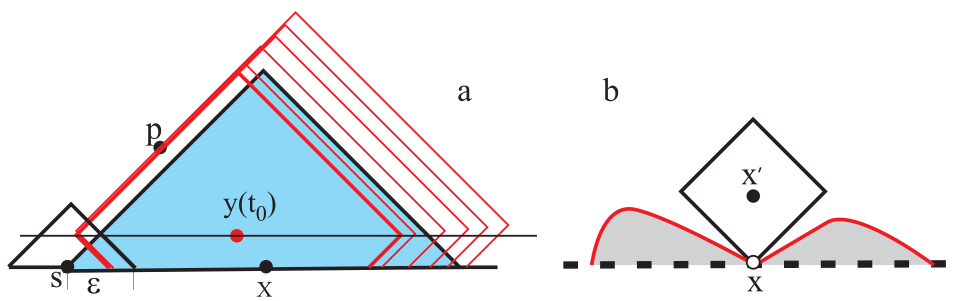

FIGURE 14. Čebyšev sets in the upper half plane U

Finally, we show that $\overline{S}_t \subset U$. Let $\mathbf{x} \in \overline{S}_t \cap B$, and let $\mathbf{x}' > \mathbf{x}$. Then $d(\mathbf{x}', S) = d(\mathbf{x}', \overline{S}) = x_2'$; but the only point of $\overline{S}$ at which this distance is achieved, $\mathbf{x}$, is not in S. (Figure 14b). ■

Corollary 4. *Any closed strongly nested arc of $\mathcal{K}_0^1$ is Čebyšev , while an antinested arc of $\mathcal{K}_0^1$ is Čebyšev if and only if it is closed in $\mathcal{K}^1$.*

Example 4. The arcs $A := \{[-t, 3t] : t \geq 0\}$ and $A' := \{[\frac{1}{2}t, \frac{3}{2}t] : t \geq 0\}$ are both Čebyšev in $\mathcal{K}^1$ (Figure 15.) The arc A is strongly nested, and $\xi_A([p,q])$ is the element $[\frac{-1}{4}(q-p), \frac{3}{4}(q-p)]$ of A with the same width (*cf* Theorem 2.5 of [7]). The arc A' is antinested, and $\xi_{A'}([p,q])$ is the element $[\frac{1}{4}(p+q), \frac{3}{4}(p+q)]$ of A' sharing the same center if there is one such, and otherwise $\{0\}$(*cf* Theorem 1 of [4].) Deleting the singletons removes, along with the element $\{0\}$ of A, all other elements of the same width; so ξ_A is well-defined on $\mathcal{K}_0^1$, and the restriction of A to $\mathcal{K}_0^1$ is Čebyšev . However, $\mathcal{K}_0^1$ contains nondegenerate intervals (such as the interval $I = [-1, 0]$ shown) that are mapped to $\{0\}$ by ξ_A; so the restriction of A' to $\mathcal{K}_0^1$ is not Čebyšev.

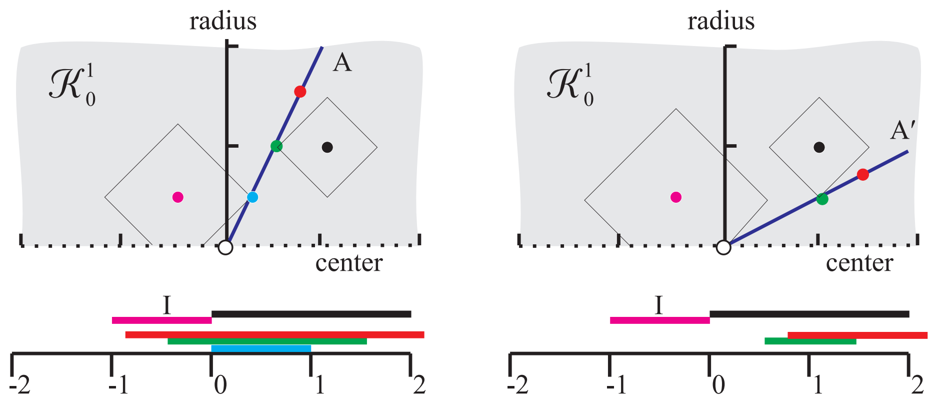

FIGURE 15. Čebyšev and non-Čebyšev arcs in $\mathcal{K}_0^1$

Observation 2. Another important class of arcs in a metric space is the class of *geodesics*: an arc is a geodesic if it is a (not necessarily "the") shortest path between any two points on it. One might wonder whether Čebyšev arcs in $\mathcal{K}^1$ are the same thing as geodesics; it is easy to show that they are not.

For instance, the polygonal arc joining the points $(-1, 0)$, $(0, -\frac{1}{2})$, and $(1, 0)$ is Čebyšev , but is not the shortest path between its endpoints. Conversely, a segment with a 45° slope is a geodesic but not Čebyšev . In fact, it may be shown (we won't) that the set of geodesic arcs in $\mathbf{R}^2_\diamond$ is the closure of the set of Čebyšev arcs in $\mathbf{R}^2_\square$ and *vice versa*.

3. Directions for further research.

This paper was motivated by the more difficult problems in higher dimensional spaces; the natural next step is to attempt to extend the results obtained here to those spaces. One could try to generalize its results in at least four different ways. In general, for $n > 1$, $\mathcal{K}^n$ and $\mathcal{O}^n$ are nonisomorphic, and neither is isomorphic to any finite-dimensional space; moreover, for $n > 2$, the "taxicab n-space" $\mathbf{R}^n_\diamond$ and the "ℓ^∞ n-space" $\mathbf{R}^n_\square$ are different. (For instance, the unit ball of $\mathbf{R}^3_\diamond$ is an octahedron, that of $\mathbf{R}^3_\square$ is a cube. The isomorphism between "square and diamond Shreddies" does not extend to higher dimensions!) Other Minkowski spaces with polyhedral balls may also be worth examining.

Interesting questions include:

Complete classification of Čebyšev sets:: For higher-dimensional hyperspaces, this has been the topic of various papers by the first author and others, and is probably quite difficult. The problem for Minkowski spaces is probably easier, and may have applications in computational geometry.

Treelike or fractal Čebyšev sets:: It is not hard to show (using Theorem 1) that no set homeomorphic to the union of three line segments with a common endpoint can be Čebyšev in $\mathbf{R}^2_\diamond$, and thus that any Čebyšev tree is an arc. Is this true in all Minkowski spaces? And do there exist Čebyšev fractals (by any interesting definition) in Minkowski spaces?

Čebyšev sets with variant dimension:: We have seen (eg, Figure 10c) that a Čebyšev set in $\mathbf{R}^2_\diamond$ may be two-dimensional in some neighborhoods and one-dimensional in others. (A convex set in a linear space cannot do this!) It seems fairly clear that this can happen in $\mathbf{R}^n_\diamond$ for $n > 2$; can it also happen in $\mathcal{K}^n$, $n > 2$?

Acknowledgements. R. Dawson was supported by an NSERC Discovery Grant and S. Levy was partially supported by an NSERC USRA.

REFERENCES

[1] Benson,R. V., *Euclidean Geometry and Convexity*, McGraw Hill (1966)
[2] A. Bogdewicz,R. Dawson, and M. Moszyńska, *Čebyšev sets in hyperspaces over a Minkowski space*, Glas. Mat., 42, No.1 (2007), 57-67
[3] A. Bogdewicz and M. Moszyńska, *Čebyšev sets in the space of convex bodies*, Rend. Circ. Mat. Palermo (2), Suppl. 77 (2006), 19-39.
[4] R. Dawson, *Some Čebyšev sets with parametric dimension* $d+1$ *in hyperspaces*, in *Convex and Fractal Geometry*, Banach Center Publications **84**(2009), 89-110
[5] R. Dawson, "Dimension of Čebyšev sets in hyperspaces" (Open Problem session), in *Convex and Fractal Geometry*, Banach Center Publications **84**(2009), 217
[6] R. Dawson, *Monotone and Čebyšev arcs in hyperspaces*, to appear, J. Geom.
[7] R. Dawson and M. Moszyńska, *Čebyšev sets in hyperspaces*, Can. Jour. Math. **61** (2009), 299-314
[8] V. Klee, Convexity of Čebyšev sets, Math. Ann. 142 (1961), 292-304.
[9] E. F. Krause, *Taxicab Geometry: An Adventure in Non-Euclidean Geometry*, Dover, New York (1975)
[10] M. Moszyńska, *Selected Topics in Convex Geometry*, Birkhäuser, Boston (2006)
[11] F. A. Valentine, *Convex Sets*, McGraw-Hill, New York (1964)
[12] "Shreddies", *Wikipedia*, n.d. Web, Aug. 9, 2010.

E-mail address: rdawson@cs.stmarys.ca scott_levy_smu@hotmail.com