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ADJACENCY PROPERTIES OF GRAPHS AND A CONJECTURE OF ERDŐS

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Abstract. In 1963, Erdős and Rényi gave a non-explicit, randomized construction of graphs with an adjacency property. For a positive integer n, a graph is *n-existentially closed* (or *n-e.c.*) if for all disjoint sets of vertices A and B with $|A \cup B| = n$, there is a vertex z not in $A \cup B$ joined to each vertex of A and no vertex of B. Until recently, only a few explicit families of n-e.c. graphs were known, such as Paley graphs. Erdős posed a conjecture on the asymptotic minimum order of n-e.c. graphs which is only now receiving attention. The exact minimum orders of n-e.c. graphs are only known for $n = 1$ and $n = 2$.

Using a computer search, a new example of a 3-e.c. graph of order 30 is presented. Previously, no known 3-e.c. graph was known to exist of that order. We give a new randomized construction of vertex-transitive n-e.c. graphs, exploiting Cayley graphs.

1. **Introduction.** The probabilistic method was discovered by Paul Erdős in the 1950's and has since found widespread use in many fields of Mathematics and Computer Science, and especially so in graph theory. The method shows objects with prescribed properties occur with positive probability, and therefore such objects exist. Finding explicit examples of the objects (in our case, graphs with an adjacency property) is often quite challenging.

An *adjacency property* is a global property of a graph, where given a fixed subset of vertices S, there exists vertices outside of S joined to vertices of S in a predetermined way. Adjacency properties stem from a seminal paper on random graphs by Erdős and Rényi [12] published in 1963. One particular adjacency property that has received much recent attention is the n-e.c. property. For a positive integer n, a graph is *n-existentially closed* (or *n-e.c.*) if for all disjoint sets of vertices A and B with $|A \cup B| = n$ (one of A or B can be empty), there is a vertex z not in $A \cup B$ joined to each vertex of A and no vertex of B. We say that z is *correctly joined* (or *c.j.*) to A and B. See Figure 1. Hence, for all n-subsets

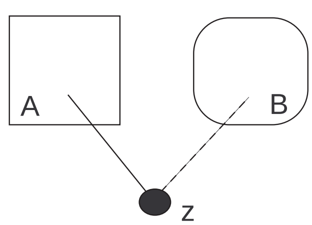

Figure 1. The n-e.c. property.

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S of vertices, there exist 2^n-many vertices joined to S in all possible ways. Although the n-e.c. property is straightforward to define, it is not obvious from the definition that graphs with the property exist.

Erdős and Rényi gave a non-explicit, randomized construction of such a graph in [12]. Roughly speaking, random graphs arise by choosing edges among pairs of distinct vertices independently with a given probability. To be more precise, the *random graph* $G(m,p)$ consists of the probability space $(\mathcal{G}_m, \mathcal{F}, \mathbb{P})$, where $\mathcal{G}_m$ is the set of all graphs with vertex set $\{1, 2, \ldots, m\}$, $\mathcal{F}$ is the family of all subsets of $\mathcal{G}_m$. Each graph is chosen independently with probability a fixed $p \in (0,1)$. There are $|\mathcal{G}_m| = 2^{\binom{m}{2}}$ graphs of order m, so the probability function is given by

$$\mathbb{P}(G) = p^{|E(G)|}(1-p)^{\binom{m}{2}-|E(G)|}$$

This space may be viewed as $\binom{m}{2}$ independent coin flips, one for each pair of vertices where the probability of drawing an edge is equal to p.

We say that an event holds *asymptotically almost surely* (a.a.s.) in $G(m,p)$ if it holds with probability tending to 1 as $m \to \infty$. We will consider asymptotic results on probability spaces such as $G(m,p)$, so we recall asymptotic notation. Let f and g be functions whose domain is some fixed subset of $\mathbb{R}$. We write $f \in O(g)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

exists and is finite. We abuse notation and write $f = O(g)$. We write $f = \Omega(g)$ if $g = O(f)$, and $f = \Theta(g)$ if $f = O(g)$ and $f = \Omega(g)$. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,$$

then $f = o(g)$. So if $f = o(1)$, then f tends to 0. We write $f \sim g$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$. All logarithms are in base e, and are written as $\log x$.

Theorem 1. *Fix an integer $n \in \mathbb{N}$. The following then holds.*

1. *A.a.s. $G(m, \frac{1}{2})$ is n-e.c.*
2. *Let f be a positive real-valued function defined by*

$$f(m,n) = \binom{m}{n} 2^n \left(1 - \frac{1}{2^n}\right)^{m-n}.$$

If m is an integer chosen so that $f(m,n) < 1$, then there is an n-e.c. graph of order m.

Proof. Let $G = G(m, 1/2)$. For item (1), let A, B be two sets of vertices of G such that $A \cap B = \emptyset$ and $|A \cup B| = n$. The probability that no vertex of G is joined correctly to A and B is

$$\left(1 - \frac{1}{2^n}\right)^{m-n}. \tag{1}$$

There are $\binom{m}{n}$ choices of an n-set of vertices X, and 2^n many partitions of X into sets A and B. Hence, by (1) it follows that the probability that G is not n-e.c. is at most

$$\begin{aligned}\binom{m}{n} 2^n \left(1 - \frac{1}{2^n}\right)^{m-n} &\leq m^n 2^n \left(1 - \frac{1}{2^n}\right)^{m-n} \\ &= \exp\left(n \log m + n \log 2 + (m-n) \log\left(1 - \frac{1}{2^n}\right)\right) \\ &= o(1),\end{aligned}$$

where the last equality follows since $\log\left(1 - \frac{1}{2^n}\right)$ is a negative constant.

For the proof of item (2), if m has the given property, then with positive probability, $G(m, \frac{1}{2})$ contains a n-e.c. graph with positive probability. □

We note that Theorem 1 easily generalizes to $G(m, p)$, where $p \in (0, 1)$ is fixed. We now consider how large m must be for $G(m, \frac{1}{2})$ to be n-e.c. with positive probability. A plot of n versus m (where the corresponding m value is found by solving the equation $\lceil f(m, n)\rceil = 1$) is shown in Figure 2. Note how quickly the function increases with n.

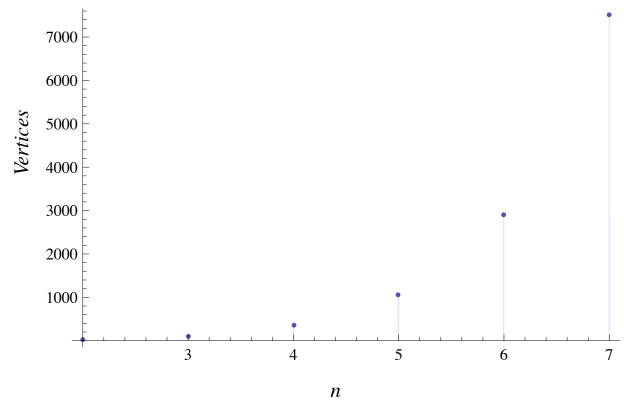

FIGURE 2. Number of vertices m needed for $G(m, 1/2)$ to be n-e.c.

We now turn our attention to the minimum order of an n-e.c. graph. For a positive integer n, denote the minimum order of an n-e.c. graph by $m_{ec}(n)$. By Theorem 1 (2), n-e.c. graphs exist for all $n > 0$, and so the function $m_{ec}(n)$ is well-defined. It was determined in [8] that $m_{ec}(1) = 4$ and $m_{ec}(2) = 9$. These are the only two known values of this function! In [8], it was shown that there are exactly three non-isomorphic 1-e.c. graphs of order four; see Figure 3.

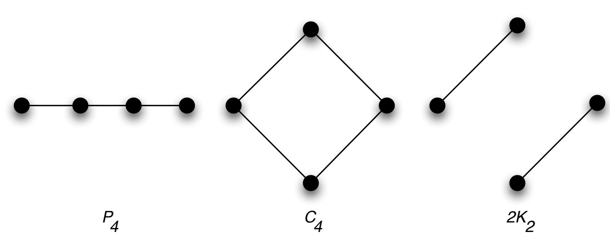

FIGURE 3. The 1-e.c. graphs of minimum order.

The unique isomorphism type of minimum order 2-e.c. graphs (as proved in [8]) is shown in Figure 4.

We note that Theorem 1 (2) supplies an asymptotic upper bound for $G(m, \frac{1}{2})$ to be n-e.c., which we describe in our next theorem. As the proof is part of folklore, but not evident in the literature, so we give it explicitly here.

Theorem 2. *If $m = O(n^2 2^n)$ and n is a sufficiently large integer, then with positive probability $G(m, \frac{1}{2})$ is n-e.c. In particular,*

$$m_{ec}(n) = O(n^2 2^n).$$

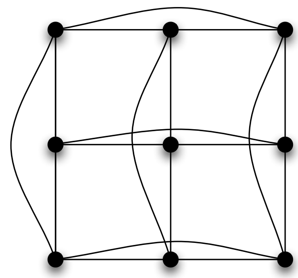

FIGURE 4. The unique 2-e.c. graph of minimum order.

Proof. We must show that if $m = O(n^2 2^n)$, then $f(m, n) < 1$. Equivalently, we show that if $\epsilon > 0$ is fixed and $m = \lfloor(\epsilon + 1)n^2 2^n\rfloor$, then

$$\log f(m) < 0. \tag{2}$$

For simplicity, we drop the floor and work with $m = (\epsilon + 1)n^2 2^n$. Now

$$\binom{m}{n} 2^n \left(1 - \frac{1}{2^n}\right)^{m-n} < m^n 2^n \left(1 - \frac{1}{2^n}\right)^{m-n}.$$

Hence, (2) is equivalent to showing that

$$n \log m + n \log 2 + (m - n) \log \left(1 - \frac{1}{2^n}\right) < 0. \tag{3}$$

For n sufficiently large we have that $\log\left(1 - \frac{1}{2^n}\right) \sim -\frac{1}{2^n}$. By this fact, by computation, and by the choice of m, (3) is equivalent to

$$n(\log(\epsilon + 1) + 2 \log n + \log 2) + n^2 \log 2 + \frac{n}{2^n} < (\epsilon + 1)n^2,$$

which is valid for large n as $\log 2 < 1$. □

The determination of $m_{ec}(n)$, where $n \geq 3$ appears to be an extremely difficult open problem. An asymptotic lower bound was proved in Erdős et al. [10], where it was shown that

$$m_{ec}(n) = \Omega(n2^n).$$

One of the deepest (and one we think deserves to be better known) conjectures on n-e.c. graphs was posed by Caccetta, Erdős, and Vijayan [10].

Conjecture 1.

$$m_{ec}(n) = \Theta(n2^n).$$

Hence, to prove the conjecture, we would need to present a family of n-e.c. graphs with order $O(n2^n)$. Theorem 2 actually gives the best known upper bound to $m_{ec}(n)$. We therefore need to reduce the order $O(n^2 2^n)$ by a multiplicative factor of n.

There has been much research done in determining the minimum order of a 3-e.c. graph. The results of [8] show that

$$20 \leq m_{ec}(3) \leq 28.$$

A new lower bound on the order of $m_{ec}(3)$ was found recently using more sophisticated computational methods. Based on 15,000 hours of CPU time, the authors of [14] demonstrated that $m_{ec}(3) \geq 24$. Hence, $m_{ec}(3)$ can only be one of

$$24, 25, 26, 27, 28.$$

We do not solve Erdős' conjecture here, nor determine $m_{ec}(3)$. However, we present some new computational results on 3-e.c. graphs of order 30 in Section 2, and a new construction of vertex-transitive n-e.c. graphs arising from random Cayley graphs in Section 3. Both results were first presented in the second author's Master's thesis [11].

1.1. **Explicit Constructions of n-e.c. Graphs.** The first family of explicit graphs that were discovered to contain n-e.c. graphs for all n were Paley graphs. Fix q a prime power with $q \equiv 1 \pmod 4$. A *Paley graph*, written P_q, is a graph constructed on the points of a finite field of order q such that two vertices are adjacent if and only if their difference is a non-zero square in the field. As an example, consider the graph P_9, which is isomorphic to the minimum order 2-e.c. graph. Let S be the set of all non-zero squares in the finite field with 9 elements, written $GF(9)$. We use the following representation of elements of this field:

$$GF(9) = \{a + bi : a, b \in \mathbb{Z}_3, i^2 = -1\}.$$

The non-zero squares are therefore $\{1, 2, i, 2i\}$. See Figure 5, where vertices are labelled by the elements of $GF(9)$.

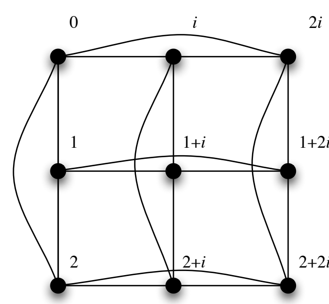

FIGURE 5. The graph P_9.

The following result on the n-e.c. properties of Paley graphs was proven independently in [4, 6]. The proof uses a famous result from number theory: Weil's proof of the Riemann hypothesis over finite fields.

Theorem 3. *If*

$$q > n^2 2^{2n-2},$$

then P_q is n-e.c.

For a more detailed discussion of construction of n-e.c. graphs arising from number theory, designs, geometry, and set theory, the reader is directed to the recent survey [7].

2. **Computational results.** The difficulty of determining if a graph is n-e.c. increases dramatically with n. For example, to check that a graph of order m is n-e.c., for each of the $\binom{m}{n}$ subsets S of vertices, we would need to find 2^n vertices joined to S in all the possible ways. This becomes difficult, if not impossible, to do by hand for large n. The focus of this section is on computational results related to orders of small 3-e.c. graphs. We recall from the introduction that

$$24 \le m_{ec}(3) \le 28$$

(the lower bound follows from [14], while the upper bound follows from [8]). We note that most of the known explicit n-e.c. graphs are strongly regular. For $n = 3$, in [2] it was shown that the Paley graph of order 29 is the minimum order 3-e.c. Paley graph. Few examples of strongly regular non-Paley n-e.c. graphs are known.

A graph G is *vertex-transitive* if any two distinct vertices of G, there is an automorphism mapping one to the other. Hence, in a vertex-transitive graph, any two vertices behave identically. For example, the graph P_9 is vertex-transitive, as is any cycle or clique.

Most of the known explicit n-e.c. graphs are strongly regular. Let k, $v > 0$, λ, and μ be non-negative integers. A k-regular graph G with v vertices, so that each pair of joined vertices has exactly λ common neighbours, and each pair of non-joined vertices has exactly μ common neighbours, is called a *strongly regular graph*; we say that G is $\mathrm{SRG}(v, k, \lambda, \mu)$. Paley graphs are an important instance of strongly regular graphs; the graph P_q is a $\mathrm{SRG}(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$ (for a proof, see [5]). In our computer search for a minimum order 3-e.c. graph, we focused on strongly regular graphs and the class of vertex-transitive graphs with orders between 24 to 30 (inclusive).

Lists containing all isotypes of small order vertex-transitive and strong-ly regular graphs are publicly available on-line. The data sets for the class of strongly regular graphs can be found in [17], while the data for the class of vertex-transitive graphs is available on-line at [16]. The data set is partitioned into different files based on the order of the graph. Each file consists of adjacency matrices encoded in the $g6$ format. (More on this format can be found at [16].) The search was conducted only on graphs of order 24 to 30 to determine if a minimum order 3-e.c. graph lies in one of these two classes. We note that the vertex-transitive graphs of orders 20 to 28 were checked for the 3-e.c. property in [8]. Although we did not determine the order of minimum order 3-e.c. graph, we found other results which we now report.

The following table summarizes the results of the computer search for 3-e.c. graphs. The numbers in the second and third columns represent the number of isomorphism types of graphs which are 3-e.c. The time required to check all the isotypes is presented along with number of isotypes checked for each order.

Order	**Vertex-Transitive**	**SRG**	**Isotypes**	**CPU hrs**
24	0	0	15506	8
25	0	0	464	0.29
26	0	0	4236	3.06
27	0	0	1434	1.16
28	2	0	25850	23.52
29	1	1	1182	1.19
30	2	0	46308	52

Before we discuss the results, we mention the numerical location of these graphs within the data sets. Hence, the numbers in the last two columns correspond to the positive integer assigned to the graphs.

Order	**Vertex-Transitive**	**Strongly Regular**
28	11440 and 15880	-
29	653	41
30	19022 and 24918	-

As $m_{ec}(3) \geq 24$, our results show that there are no 3-e.c. strongly regular or vertex-transitive graphs of order less than 28. The two 3-e.c. graphs of order 28 (first found in [8]) are not isomorphic and one is the complement of the other. The adjacency matrix is shown below.

$$\begin{pmatrix}
0&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&1&1&0&0&0&0&1&0&1&1&1&0&0&1&1&0&0&0&1&1&0&1&0&0&0&1\\
1&1&0&0&0&1&1&0&0&1&1&0&1&0&1&1&0&1&0&1&0&0&0&0&0&1&1&0\\
1&1&0&0&1&1&0&0&0&0&1&0&1&1&1&0&1&0&1&1&0&0&1&0&0&1&0&0\\
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0&1&0&0&0&1&0&1&1&0&0&0&0&0&1&1&1&1&1&0&1&0&1&1&0&0&1&0
\end{pmatrix}$$

It can be shown that the 3-e.c. graph of order 29 found through our search is isomorphic to P_{29}, the Paley graph of order 29. By deleting nodes of P_{29} and then adding edges to the resulting graph, we found several non-isomorphic 3-e.c. graphs of order 28.

In [15] it was mentioned that the existence of 3-e.c. graphs of order 30 is unknown. The 3-e.c. graphs of order 30 found through our search settles this problem. The two 3-e.c. graphs we found of order 30 are complements of each other. (This can be verified using *Mathematica*.) Below we present the adjacency matrix of one of these graphs.

$$\begin{pmatrix}
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1&0&0&0&0&0&0&1&1&1&1&0&0&0&0\\
1&0&0&1&1&1&1&0&0&1&1&0&0&1&1&0&1&0&0&0&1&1&0&0&1&1&0&0&0&0\\
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0&0&0&0&1&1&1&1&0&0&1&0&1&1&0&0&0&1&1&0&0&1&0&1&0&0&0&1&1&1\\
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0&0&1&1&1&0&0&1&0&1&0&1&0&0&1&0&0&0&0&1&1&0&1&0&0&1&1&1&0&1\\
0&1&0&0&0&1&0&1&1&0&1&0&1&1&0&0&0&0&1&0&0&1&0&1&1&1&1&0&0&1\\
0&1&0&0&0&0&1&1&1&1&0&1&0&0&1&0&0&0&0&1&1&0&1&0&1&1&0&1&1&0\\
0&1&1&0&0&0&0&0&1&0&1&1&0&1&0&1&1&1&0&0&1&0&1&1&0&0&0&1&0&1\\
0&1&1&0&0&0&0&1&0&1&0&0&1&0&1&1&1&1&0&0&0&1&1&1&0&0&1&0&1&0\\
0&0&0&1&0&0&1&0&1&0&0&1&1&0&1&0&1&0&1&0&1&1&1&0&0&1&0&1&0&1\\
0&0&0&0&1&1&0&1&0&0&0&1&1&1&0&0&1&0&0&1&1&1&0&1&1&0&1&0&1&0\\
0&0&0&1&0&0&1&0&1&1&0&0&0&1&1&1&0&0&1&1&1&0&0&1&0&1&0&1&0&1\\
0&0&0&0&1&1&0&1&0&0&1&0&0&1&1&1&0&0&1&1&0&1&1&0&1&0&1&0&1&0
\end{pmatrix}$$

3. **Cayley graphs and vertex-transitive n-e.c. graphs.** We consider a construction (albeit a randomized one) of n-e.c. graphs using Cayley graphs. The novel feature of the graphs we generate is that they are always vertex-transitive, unlike the n-e.c. graphs arising from $G(m,p)$. We recall the definition of Cayley graphs. Given a group H, let S be a non-empty subset of H that is closed with respect to taking inverses, and does not contain the identity element e. The set S is called the *connection set*. The *Cayley graph*, denoted by $H(S)$, has vertices the elements of H, and $x, y \in E(H(S))$ if and only if $xy^{-1} \in S$. Cayley graphs are an important class of vertex-transitive graphs (see [13], for example). Hence, $H(S)$ is a regular graph.

Given a group H, we consider a way of randomly choosing the connection set S. We begin by defining a set S' to contain all the pairs (g, g^{-1}) from H, except for the pair (e, e). Fix a real number $p \in (0,1)$. For each pair $(g, g^{-1}) \in S'$, elements g, g^{-1} are added independently and with probability p to S; with probability $(1-p)$, g, g^{-1} is not added to S. We note that S is a well-defined connection set since it is inverse-closed, and it does not contain the identity element. We name the corresponding probability space *the random Cayley graphs on the group H of order m with probability p* and write $\mathcal{H}_m(p)$. Random Cayley graphs have been studied especially in their connection with expansion properties of graphs; see [1].

While $|S|$ is a random variable in $\mathcal{H}_m(p)$, all choices of S give rise to Cayley graphs, and hence, vertex-transitive graphs. This follows directly from the definition of Cayley graphs. We prove the following result in the case when $p = 1/2$. The method of proof is based on a construction of vertex-transitive n-e.c. tournaments exploiting circulant tournaments in [9].

Theorem 4. *For n a positive integer, a.a.s. $\mathcal{H}_m(1/2)$ is n-e.c.*

Proof. Consider the graph $G = \mathcal{H}_m(1/2)$ with order m. Fix

$$X = \{x_1, x_2, \ldots, x_n\}$$

an n-set of vertices of G. We need to find a vertex z correctly joined to X (regardless of the partition of X into two sets, say A and B). For $v \in G$ define

$$\sigma_X(v) = \{u \in S : \text{for some } x \in X, u = vx^{-1} \text{ or } u = xv^{-1}\}$$

We would like to show we can construct a set U, disjoint from X, such that with probability tending to 1, there is a $z \in U$ that is correctly joined to X. Equivalently, we show that with probability $o(1)$, there is no vertex in U correctly joined to X. We construct U such that $|U| = \lfloor \frac{m}{4n^2} \rfloor$, and impose the following restrictions on U.

1. For all distinct z and z' in U, $\sigma_X(z) \cap \sigma_X(z') = \emptyset$.
2. $|\sigma_X(z)| = n$.

From the definition of σ_X, it is straightforward to see that item (1) ensures the event that z is joined to a vertex x_i in X is independent of the event that z' is joined to x_i, and item (2) ensures that the events that z is joined to any particular x_i are mutually independent.

We inductively construct the set U_k whose union will be U. We choose U_1 to be a single vertex z_1 not in X with the property that $|\sigma_X(z_1)| = n$. We therefore eliminate elements in X and those z_1 such that $|\sigma_X(z_1)| < n$. For example, if it happens that $z_1 x_i^{-1} = x_j z_1^{-1}$ for some i and j, then we must eliminate z_1 from consideration. Each distinct pair of vertices from X eliminates at most one element of G. We may now find a suitable z_1 since

$$m - n - \binom{n}{2} > 0.$$

(Recall that n is a constant that does not depend on m.)

Suppose that U_k has been constructed for a fixed $k < \lfloor \frac{m}{4n^2} \rfloor$, so that $|U_k| = k$, and the set U_k has elements satisfying items (1) and (2). Set $U_k = \{z_1, \ldots, z_k\}$. We choose z_{k+1} as the new element of U_k by eliminating elements from $V(G) \setminus U_k$. As in the base step, by considering all the pairs of vertices from X, $\binom{n}{2}$ vertices are eliminated. Each vertex $z \in U_k$ satisfies $|\sigma_X z| = n$. To ensure that $\sigma_X(z) \cap \sigma_X(z') = \emptyset$ for $z \in U_k$ and $z' \in U_{k+1}$, we must eliminate another $2kn$ vertices. For large m, we may find a suitable z_{k+1} since

$$\begin{aligned} m - n - k - \binom{n}{2} - 2kn &> m\left(1 - \frac{1}{4n^2} - \frac{1}{2n}\right) - n - n^2 \\ &> 0. \end{aligned}$$

Add z_{k+1} to U_k, to form U_{k+1}. Define $U = U_{\lfloor \frac{m}{4n^2} \rfloor}$, so $|U| = \lfloor \frac{m}{4n^2} \rfloor$ as desired.

We now estimate the probability that none of the vertices of U are correctly joined to X, and show this tends to 0 as m tends to ∞. By items (1) and (2), we have that

$$\mathbb{P}(\text{no } z \text{ in } U \text{ is c.j. to } X) = \left(1 - \frac{1}{2^n}\right)^{\lfloor \frac{m}{4n^2} \rfloor}.$$

Hence, we have that the probability P of the event that G is not n-e.c. satisfies

$$\begin{aligned} P &\leq \binom{m}{n} 2^n \left(1 - \frac{1}{2^n}\right)^{\frac{m}{4n^2}} \\ &\leq m^n 2^n \left(1 - \frac{1}{2^n}\right)^{\frac{m}{4n^2}} \\ &= \exp\left(n \log m + n \log 2 + \left(\frac{m}{4n^2}\right) \log\left(1 - \frac{1}{2^n}\right)\right) \\ &= o(1). \quad \square \end{aligned}$$

We note that the proof of Theorem 4 generalizes to $p \in (0,1)$, and we may allow n to grow as a function of m. We omit these generalizations, as our more modest goal here is to provide a new randomized construction of vertex-transitive n-e.c. graphs.

The proof of Theorem 4 gives an asymptotic upper bound for $\mathcal{G}_m(1/2)$ to be n-e.c., whose proof is similar to Theorem 2 and so is omitted.

Theorem 5. *If $m = O(n^3 2^n)$ and n is a sufficiently large integer, then with positive probability $\mathcal{G}_m(1/2)$ is n-e.c. In particular, there is a vertex-transitive n-e.c. graph of order $O(n^3 2^n)$.*

4. **Future work.**

We know little about the minimum order n-e.c. graphs; for example, are they all vertex-transitive or even regular? The determination of $m_{ec}(3)$ will likely use a mixture of computational and theoretical results on 3-e.c. graphs. By the results in [15] and those referenced in Section 2, the only orders where we do not know whether a 3-e.c. graph exists are:

$$24, 25, 26, 27, 31, 33.$$

Determining the exact order of $m_{ec}(n)$ for $n \geq 4$ appears to be a difficult problem. Even determining the asymptotic order of this function presents a challenge. The conjecture of Erdős that

$$m_{ec}(n) = O(n 2^n),$$

remains as one of the deepest problems in this area of graph theory. Even a seemingly modest improvement to $O(n^{2-c} 2^n)$, where $c > 0$, would represent a breakthrough. It is possible that random graphs stemming from either combinatorial designs (see [3]) or Cayley graphs may eventually be adapted to solve the conjecture.

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