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REMARK ON THE PAPER "SHARP WELL-POSEDNESS AND ILL-POSEDNESS RESULTS FOR A QUADRATIC NON-LINEAR SCHRÖDINGER EQUATION" BY I. BEJENARU AND T. TAO

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Abstract. In the present paper, we consider the paper "*Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation*," J. Funct. Anal. **233** (2006), 228–259. The argument in it is very clever, but there seem to be a few errors in the proof. The purpose of this paper is to give a corrected proof.

1. **Introduction.** In [2] Bejenaru and Tao treated the initial value problem (IVP) of a quadratic Schrödinger equation

$$\begin{cases} iu_t + u_{xx} = u^2 \ , \quad u : \mathbb{R}\times\mathbb{R} \to \mathbb{C} \ , \\ \quad u(0,x) = u_0(x) \ , \end{cases} \tag{1}$$

and they improved the previous result for the local well-posedness of the IVP (1) established by Kenig, Ponce and Vega [9]. Since their argument seems to have wide application, the paper is worth reading for those studying well-posedness theory of nonlinear evolution equations. In this article we shall complete their proof, fixing a few correctable errors.

The nonlinear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = F(u) \ , \quad u : \mathbb{R}\times\mathbb{R}^n \to \mathbb{C} \ , \\ \quad u(0,x) = u_0(x) \end{cases} \tag{2}$$

appears in various regions of mathematical physics, so it has been extensively studied in all aspects. In particular, (2) with a power-type nonlinearity F has various good properties (e.g. scaling invariance (4)) and many results are known. We restrict our attention to the local well-posedness of the IVP. Local (resp. global) well-posedness of the IVP in a data space $\mathcal{D}$, which is one of fundamental problems for an evolution equation, basically requires existence of a solution in a time interval $[0,T)$, $T>0$ (resp. $[0,\infty)$) for all data in $\mathcal{D}$, uniqueness of the solution in a suitable function space and continuous dependence of solutions on the data. The data space $\mathcal{D}$ is usually set to the Sobolev space $H^s(\mathbb{R}^n)$, the Banach space of distributions that the norm

$$\|\varphi\|_{H^s_x} := \|\langle\xi\rangle^s\hat{\varphi}\|_{L^2_\xi}$$

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is finite. Here $\langle\cdot\rangle := (1+|\cdot|^2)^{1/2}$ and $\hat{}$ denotes the Fourier transform

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{ix\cdot\xi}\varphi(x)\,dx \ .$$

Fourier analysis plays an important role in the well-posedness theory.

The linear Schrödinger equation $iu_t + \Delta u = 0$ generates the unitary group $\{e^{it\Delta}\}_{t\in\mathbb{R}}$ on H^s. Then, as a perturbation problem, the IVP (2) is replaced with the equation of integral form by Duhamel's principle,

$$u(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-t')\Delta}F(u(t'))\,dt' \ . \qquad (3)$$
$$(\ =: \Phi(u)(t) \ )$$

The most general way to solve (3) is the iteration method, i.e. to show that Φ is a contraction on a suitable function space $\mathcal{S}$. When s is sufficiently large, we can obtain well-posedness with $\mathcal{S} = C_t([0,T], H^s_x(\mathbb{R}^n))$ for small $T>0$, but for small s the map Φ fails to be a contraction on this space and we need to restrict the domain $\mathcal{S}$ effectually. Our interest is then to obtain the lowest Sobolev regularity s for local well-posedness.

Let us consider the standard power-type nonlinearity

$$F(u) = \pm|u|^{p-1}u \ , \qquad p>1 \ .$$

Then the IVP (2) is locally well-posed in $H^s(\mathbb{R}^n)$ for $s \geq \max\{0, s_c\}$ (e.g. [4])[1], where

$$s_c := \frac{n}{2} - \frac{2}{p-1} \ .$$

The proof is based on the Strichartz estimates ([14]). Furthermore, the map from data to solutions is known not to be uniformly continuous from H^s to $C_t([0,T], H^s_x)$ for any $s<\max\{0,s_c\}$, even for small T and on a small ball including the origin (e.g. [10], [5])[2]. This negative result is natural when we consider two kinds of invariance of the equation, that is, the invariance under the scaling

$$u(t,x) \quad \mapsto \quad \lambda^{-2/(p-1)}u(\lambda^{-2}t, \lambda^{-1}x) \ , \qquad \lambda>0 \qquad (4)$$

which conserves the $\dot{H}^{s_c}(\mathbb{R}^n)$ norm, and that under the Galilean transform

$$u(t,x) \quad \mapsto \quad e^{iv\cdot x - i|v|^2 t}u(t, x-2vt) \ , \qquad v\in\mathbb{R}^n$$

which conserves the $L^2(\mathbb{R}^n)$ norm.

This invariance argument suggests that it would be possible to show the well-posedness under some negative Sobolev regularities if we choose the nonlinearity such that the equation is not Galilean invariant, and that the critical scaling regularity s_c is negative. For instance, when the nonlinearity is u^2 or $\bar{u}^2$ or $u\bar{u}$, the local well-posedness regularity threshold can be negative in spatial dimension $n=1,2,3$. In the 1D case, Kenig, Ponce and Vega [9] actually obtained the local well-posedness for these nonlinearities under negative regularity; $s>-3/4$ for $F(u)=c_1u^2+c_2\bar{u}^2$ and $s>-1/4$ for $F(u)=cu\bar{u}$ ($c_1,c_2,c\in\mathbb{C}$). Similar results are established by

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[1] We need an additional regularity assumption if p is not an odd integer.

[2] The iteration method usually provides the *uniform* continuity of the data-to-solution map, so it seems that we cannot apply this method directly to establish the well-posedness for these regularities. But the lack of uniform continuity does not necessarily imply *ill*-posedness. In fact, there are the cases where the map is continuous but not uniformly continuous (e.g. [17] for the modified Korteweg-de Vries equation).

Colliander, Delort, Kenig and Staffilani [6] for 2D, and by Tao [16] for 3D. It is interesting that the regularity threshold depends on the nonlinearity itself, while the result from the Strichartz estimates depends only on the degree of the nonlinearity. Their proofs are based on the *Fourier restriction norm* for space-time functions introduced by Bourgain [3] and defined for Schrödinger equations as

$$\|u\|_{X^{s,b}_{t,x}} := \|\langle\xi\rangle^s\langle\tau-|\xi|^2\rangle^b\tilde{u}\|_{L^2_{\tau,\xi}(\mathbb{R}^{1+n})}\ ,$$

where $\tilde{\cdot}$ denotes the space-time Fourier transform. The function space $\mathcal{S}$ is then set to $X^{s,b}$ with some $b>1/2$, which is a Banach space of distributions with the above norm and continuously embedded into $C_t(H^s_x)$. The Fourier restriction norm method has been applied to various nonlinear evolution equations (e.g. [7], [8], [15], [17], [18]).

For the iterative argument on $X^{s,b}$, the following *bilinear estimate* associated with the nonlinearity is the most important and difficult to establish[3];

$$\|B(u,v)\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}}\|v\|_{X^{s,b}}, \tag{5}$$

where $B(u,v)$ is the bilinear operator corresponding to the nonlinearity,

$$B(u,v) := \begin{cases} uv & \text{if } F(u)=u^2, \\ \bar{u}\bar{v} & \text{if } F(u)=\bar{u}^2, \\ u\bar{v} & \text{if } F(u)=u\bar{u}. \end{cases}$$

The estimate (5), however, fails when $s\le -3/4$ for $u^2,\bar{u}^2$ and $s\le -1/4$ for $u\bar{u}$ ([9], [13]). Here we should remark that the failure of the bilinear estimate in $X^{s,b}$ does *not* imply the ill-posedness at these regularities, because there remains the possibility that we can restore it using different space-time spaces of functions. We need a further modification on the function space $\mathcal{S}$ to treat lower regularities.

For 1D and the nonlinearity u^2, Bejenaru and Tao [2] subtly modified the space $X^{s,b}$ and established the local well-posedness[4] in $H^s(\mathbb{R})$, $s\ge -1$. In this situation, the crucial bilinear estimate is rewritten in the Fourier space as

$$\|\langle\tau-\xi^2\rangle^{-1}f*g\|_{\hat{\mathcal{S}}^s} \lesssim \|f\|_{\hat{\mathcal{S}}^s}\|g\|_{\hat{\mathcal{S}}^s}, \tag{6}$$

where $*$ means the space-time convolution

$$(f*g)(\tau,\xi) := \int_{\mathbb{R}}\int_{\mathbb{R}} f(\tau-\tau',\xi-\xi')g(\tau',\xi')\,d\tau'd\xi'\ ,$$

and $\hat{\mathcal{S}}^s$ should be a refinement of $\hat{X}^{s,b}$, spaces of the Fourier transform of functions in $X^{s,b}$ equipped with the norm

$$\|f\|_{\hat{X}^{s,b}_{\tau,\xi}} := \|\langle\xi\rangle^s\langle\tau-\xi^2\rangle^b f\|_{L^2_{\tau,\xi}(\mathbb{R}^2)}\ .$$

Their modification is based on the special property of the nonlinearity u^2, so their function space does not work for other nonlinearities $\bar{u}^2$, $u\bar{u}$. However, their idea for modifying the space was applied to 2D and another nonlinearity $\bar{u}^2$ ([1], [11], [12]), and it is strongly expected that there are a number of applications to other nonlinearities or other nonlinear dispersive equations.

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[3]We use the notation $x\lesssim y$ if there exists a constant $C>0$ independent of any variable appearing in the estimate, such that $x\le Cy$, and also $x\sim y$ if $x\lesssim y$ and $y\lesssim x$. We write $x\ll y$ if the estimate $x\le C^{-1}y$ holds for some large positive constant C.

[4]They obtained the local well-posedness in a weak form; there exists a continuous data-to-solution map which agrees with the standard unique solutions on smooth data.

The rest of this paper is planned as follows. In Section 2 we recall the arguments in [2] and point out two errors in the proof of (6). They are independent of each other, and other proofs of corresponding parts will be given in Section 3 and Section 4.

2. **Ideas for the bilinear estimate.** In the following, L^pL^q always denotes the mixed Lebesgue space of variable (τ,ξ);

$$L^pL^q := L^p_\xi L^q_\tau(\mathbb{R}^2)\ , \qquad \|f\|_{L^pL^q} := \Big\| \|f(\cdot,\xi)\|_{L^q_\tau(\mathbb{R})}\Big\|_{L^p_\xi(\mathbb{R})}\ ,$$

and we also use the restricted norm; for $\Omega\subset\mathbb{R}^2$,

$$\|f\|_{L^pL^q(\Omega)} := \|f\cdot \mathbf{1}_\Omega\|_{L^pL^q},$$

where $\mathbf{1}_\Omega$ denotes the characteristic function of Ω. A new function space for solutions was then constructed in [2] in the following way.

Step 1. We call a function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{C}$ *reasonable* when $f\in L^\infty_{t,x}(\mathbb{R}^2)$ and $\operatorname{supp} f$ is compact. For $s,b\in\mathbb{R}$, we define $\hat{X}^{-1,1/2,1}$ to be the Besov endpoint of $\hat{X}^{s,b}$ whose norm is defined by

$$\|f\|_{\hat{X}^{-1,1/2,1}} := \left(\sum_{j\ge 0} 2^{-2j}\left(\sum_{d\ge 0} 2^{d/2}\|f\|_{L^2L^2(A_j\cap B_d)}\right)^2\right)^{1/2},$$

where $\{A_j\}$ and $\{B_d\}$ are two dyadic decompositions of $\mathbb{R}^2$; for non-negative integers j,d,

$$\begin{aligned} A_j &:= \left\{(\tau,\xi)\ :\ 2^j\le\langle\xi\rangle<2^{j+1}\right\},\\ B_d &:= \left\{(\tau,\xi)\ :\ 2^d\le\langle\tau-\xi^2\rangle<2^{d+1}\right\}.\end{aligned}$$

We note that

$$\|f\|_{\hat{X}^{-1,1/2}} \sim \left(\sum_{j\ge 0} 2^{-2j}\sum_{d\ge 0} 2^{d}\|f\|^2_{L^2L^2(A_j\cap B_d)}\right)^{1/2}.$$

Step 2. We use an L^1_τ-based space Y defined by the norm

$$\|f\|_Y := \|\langle\xi\rangle^{-1}f\|_{L^2L^1} + \|f\|_{L^2L^2},$$

and define the sum space $Z:=\hat{X}^{-1,1/2,1}+Y$ with the norm

$$\|f\|_Z := \inf\left\{\|f_1\|_{\hat{X}^{-1,1/2,1}}+\|f_2\|_Y : f=f_1+f_2,\ f_1\in\hat{X}^{-1,1/2,1},\ f_2\in Y\right\}.$$

Step 3. We introduce a weighted space W defined by

$$\|f\|_W := \|wf\|_Z,$$

$$w(\tau,\xi)=w(\tau):=\left[\min\{-1,\tau\}\right]^{10},$$

and define the final space $\mathcal{S}$ by $\hat{\mathcal{S}}:=W$.

The bilinear estimate (6) is then equivalent to

$$\left\|\frac{w}{\langle\tau-\xi^2\rangle}\left(\frac{f}{w}*\frac{g}{w}\right)\right\|_Z \lesssim \|f\|_Z\|g\|_Z \tag{7}$$

for all test functions f, g.

We use the following notations for simplicity;

$$A_{<j_1} := \bigcup_{j<j_1} A_j \ , \quad B_{[d_1,d_2)} := \bigcup_{d\in[d_1,d_2)} B_d \ , \quad \dots \text{etc.},$$

and the first (or the second) subscript of a function means the restriction to $\{A_j\}$ (or $\{B_d\}$); for example,

$$f_{j_1} := f\cdot \mathbf{1}_{A_{j_1}} \ , \quad g_{\geq j_2,(d_1,d_2]} := g\cdot \mathbf{1}_{A_{\geq j_2}\cap B_{(d_1,d_2]}} \ , \quad \dots \text{etc.}$$

Also, we always use variables (τ_1,ξ_1) for f, (τ_2,ξ_2) for g and (τ,ξ) for $f*g$ under the convention

$$(\tau,\xi) = (\tau_1,\xi_1) + (\tau_2,\xi_2). \tag{8}$$

If (8) holds, then one of the following cases must occur:

- $|\xi| \sim |\xi_1| \gtrsim |\xi_2|$ "High-low interaction",
- $|\xi| \sim |\xi_2| \gtrsim |\xi_1|$ "Low-high interaction", and
- $|\xi| \ll |\xi_1| \sim |\xi_2|$ "High-high interaction".

We now decompose f, g and $f*g$ with respect to ξ variable,

$$\left\| \frac{w}{\langle \tau-\xi^2\rangle}\left(\frac{f}{w} * \frac{g}{w}\right)\right\|_Z = \left\| \sum_{j,j_1,j_2\geq 0} \mathbf{1}_{A_j} \frac{w}{\langle \tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w} * \frac{g_{j_2}}{w}\right)\right\|_Z ,$$

then in order for the inner summand to be non-zero one of the following must hold:

- $|j_1 - j| \leq 10$ and $j_2 \leq j_1 + 11$ (High-low interaction),
- $|j_2 - j| \leq 10$ and $j_1 \leq j_2 + 11$ (Low-high interaction), and
- $|j_1 - j_2| \leq 1$ and $j < j_1 - 10, j_2 - 10$ (High-high interaction).

The former two cases are symmetric, so in order to prove (7) it suffices to verify

- $$\left\| \sum_{\substack{j,j_1,j_2\geq 0\\ |j_1-j|\leq 10,\ j_2\leq j_1+11}} \mathbf{1}_{A_j} \frac{w}{\langle \tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w} * \frac{g_{j_2}}{w}\right)\right\|_Z \lesssim \|f\|_Z\|g\|_Z, \text{ and}$$
- $$\left\| \sum_{\substack{j,j_1,j_2\geq 0\\ |j_1-j_2|\leq 1,\ j<j_1-10}} \mathbf{1}_{A_j} \frac{w}{\langle \tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w} * \frac{g_{j_2}}{w}\right)\right\|_Z \lesssim \|f\|_Z\|g\|_Z.$$

From Schur's lemma, the bilinear estimate (7) is finally reduced to the following.

- If non-negative integers j, j_1, j_2 satisfy $|j_1 - j| \leq 10$ and $j_2 \leq j_1 + 11$, then
$$\left\| \mathbf{1}_{A_j} \langle \tau-\xi^2\rangle^{-1} f_{j_1} * g_{j_2}\right\|_Z \lesssim \left(2^{-\delta j_2} + 2^{-\delta(j-j_2)}\right) \|f_{j_1}\|_Z \|g_{j_2}\|_Z \tag{9}$$
holds for some $\delta > 0$. (High-low interaction estimate)
- If non-negative integers j_1, j_2 satisfy $|j_1 - j_2| \leq 1$, then
$$\left\| \mathbf{1}_{A_{<j_1-10}} \frac{w}{\langle \tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w} * \frac{g_{j_2}}{w}\right)\right\|_Z \lesssim \|f_{j_1}\|_Z \|g_{j_2}\|_Z \tag{10}$$
holds. (High-high interaction estimate)

Note that the weight w satisfies $w(\tau,\xi) \lesssim w(\tau_1,\xi_1)w(\tau_2,\xi_2)$ whenever (8) holds, so we have the estimate

$$\left\| \frac{w}{\langle \tau-\xi^2\rangle}\left(\frac{f}{w} * \frac{g}{w}\right)\right\|_Z \lesssim \left\| \langle \tau-\xi^2\rangle^{-1} f * g\right\|_Z . \tag{11}$$

We have already used (11) in deriving (9).

The following lemmas and propositions were used in [2] to prove (9) and (10)[5].

Lemma 1. *Assume* supp $f \subset B_{\geq d}$ *for some* $d \geq 0$. *Then for* $b > 1/2$ *we have*

$$\|f\|_{\hat{X}^{-1,1/2,1}} \lesssim 2^{-d(b-1/2)} \|f\|_{\hat{X}^{-1,b}}.$$

Proposition 4. *Assume* supp $f \subset A_j \cap B_{\geq d}$ *for some* $j, d \geq 0$. *Then we have*

$$\|f\|_{L^2L^1} \lesssim 2^j \|f\|_Z, \tag{12}$$

$$\|f\|_{L^2L^2} \lesssim (1 + 2^j 2^{-d/2}) \|f\|_Z, \qquad \|f\|_{L^2L^2} \lesssim \|f\|_Y, \tag{13}$$

$$\|f\|_{L^1L^1} \lesssim 2^{3j/2} \|f\|_Z, \tag{14}$$

$$\|f\|_{L^1L^2} \lesssim 2^{j/2}(1 + 2^j 2^{-d/2}) \|f\|_Z, \qquad \|f\|_{L^1L^2} \lesssim 2^{j/2} \|f\|_Y. \tag{15}$$

Lemma 2. *Assume* supp $f \subset \bigcup_j (A_j \cap B_{\leq 2j+100})$. *Then we have*

$$\|f\|_{\hat{X}^{-1,1/2,1}} \sim \|f\|_Z.$$

Lemma 3. *For any test functions* f *and* g, *we have*

$$\left\| \frac{w}{\langle \tau - \xi^2 \rangle} \left(\frac{f}{w} * \frac{g}{w} \right) \right\|_Z \lesssim \|f\|_Y \|g\|_Y.$$

Proposition 5. *Suppose that* supp $f \subset A_{j_1}$, supp $g \subset A_{j_2}$ *for some* $j_1, j_2 \geq 0$, *and that there is* $D \geq 0$ *such that* $|\xi_1 - \xi_2| \geq D$ *whenever* $(\tau_1, \xi_1) \in$ supp f *and* $(\tau_2, \xi_2) \in$ supp g. *Then*

$$\|f * g\|_{L^2L^2} \lesssim 2^{j_1+j_2} \langle D \rangle^{-1/2} \|f\|_{\hat{X}^{-1,1/2,1}} \|g\|_{\hat{X}^{-1,1/2,1}}.$$

Corollary 1. *Suppose that* supp $f \subset A_{j_1}$, $\Omega \subset A_j$ *for some* $j_1, j \geq 0$, *and that there is* $D \geq 0$ *such that* $|\xi_1 + \xi| \geq D$ *whenever* $(\tau_1, \xi_1) \in$ supp f *and* $(\tau, \xi) \in \Omega$. *Then, for any* $d \geq 0$ *and any test function* g, *we have*

$$2^{-d/2} \|f * g\|_{L^2L^2(\Omega \cap B_d)} \lesssim 2^{j_1} (2^{d/2} + D)^{-1/2} \|f\|_{\hat{X}^{-1,1/2,1}} \|g\|_{L^2L^2}.$$

However, it seems that their proofs for Lemma 3 and Corollary 1 are incomplete. The error in Lemma 3 is somewhat serious[6]. On the other hand, the error in Corollary 1 seems a rather minor one, and the correction is straightforward (but needs some more efforts). We shall consider fixing these errors and give a proof of (9) and (10) without these two lemmas.

3. **Proof for the first part.**

We now consider Lemma 3. The proof of this lemma in [2] was very simple:

> (11), *Young's inequality and* (13) *imply the estimate*
>
> $$\begin{aligned} \left\| \frac{w}{\langle \tau - \xi^2 \rangle} \left(\frac{f}{w} * \frac{g}{w} \right) \right\|_Z &\lesssim \left\| \langle \tau - \xi^2 \rangle^{-1} f * g \right\|_{\hat{X}^{-1,1/2,1}} \\ &\leq \sup_{(\tau,\xi)} \left| (f * g)(\tau, \xi) \right| \cdot \left\| \langle \tau - \xi^2 \rangle^{-1} \right\|_{\hat{X}^{-1,1/2,1}} \\ &\leq \|f\|_Y \|g\|_Y \left\| \langle \tau - \xi^2 \rangle^{-1} \right\|_{\hat{X}^{-1,1/2,1}}, \end{aligned}$$
>
> *and it is easily verified that* $\left\| \langle \tau - \xi^2 \rangle^{-1} \right\|_{\hat{X}^{-1,1/2,1}} \leq C$. □

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[5]We use the same numbers of lemmas as those in [2], while new ones are indicated by "Ex."

[6]Later, Bejenaru and Tao gave a complete proof of Lemma 3; see the revised version of their paper `arXiv:0508210v4`. Our approach is different from theirs, because it is based on the boundedness of the Hilbert transform.

But, in fact, we have

$$\begin{aligned}\left\|\langle\tau-\xi^2\rangle^{-1}\right\|_{\hat{X}^{-1,1/2,1}} &= \left\{\sum_{j\ge 0} 2^{-2j}\left(\sum_{d\ge 0} 2^{d/2}\left\|\langle\tau-\xi^2\rangle^{-1}\right\|_{L^2L^2(A_j\cap B_d)}\right)^2\right\}^{1/2}\\ &\sim \left\{\sum_{j\ge 0} 2^{-2j}\left(\sum_{d\ge 0} 2^{d/2}\cdot 2^{-d}\cdot (2^{j+d})^{1/2}\right)^2\right\}^{1/2}\\ &\sim \left\{\sum_{j\ge 0} 2^{-j}\left(\sum_{d\ge 0} 1\right)^2\right\}^{1/2}\\ &= +\infty.\end{aligned}$$

Therefore Lemma 3 has not been proved. This lemma was used only to prove one restricted case of (10):

> *Suppose* $|j_1-j_2|\le 1$, supp $f_{j_1}\subset A_{j_1}\cap B_{\ge 2j_1-100}$ *and* supp $g_{j_2}\subset A_{j_2}\cap B_{\ge 2j_2-100}$. *Then,*
>
> $$\left\|\mathbf{1}_{A_{<j_1-10}\cap B_{<2j_1-10}}\frac{w}{\langle\tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w}*\frac{g_{j_2}}{w}\right)\right\|_Z \lesssim \|f_{j_1}\|_Z\,\|g_{j_2}\|_Z\,. \tag{16}$$

Now, we try to prove (16) without using Lemma 3. In this case, from (13) we may measure f_{j_1} and g_{j_2} in L^2L^2 instead of in Z. First, we divide the L.H.S. of (16) and use (11) to have

$$\begin{aligned}&\left\|\mathbf{1}_{A_{<j_1-10}\cap B_{<2j_1-10}}\frac{w}{\langle\tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w}*\frac{g_{j_2}}{w}\right)\right\|_Z\\ &\quad\lesssim \left\|\mathbf{1}_{D_1}\langle\tau-\xi^2\rangle^{-1}f_{j_1}*g_{j_2}\right\|_{\hat{X}^{-1,1/2,1}} + \left\|\mathbf{1}_{D_2}\frac{w}{\langle\tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w}*\frac{g_{j_2}}{w}\right)\right\|_Y,\end{aligned}$$

where

$$D_1 := \bigcup_{j<j_1-10}\left(A_j\cap B_{<2j+5}\right)\,,\quad D_2 := \bigcup_{j<j_1-10}\left(A_j\cap B_{[2j+5,2j_1-10)}\right)\,.$$

To estimate the first term, we can follow the argument in the proof of Lemma 3, because

$$\begin{aligned}\left\|\mathbf{1}_{D_1}\langle\tau-\xi^2\rangle^{-1}\right\|_{\hat{X}^{-1,1/2,1}} &\sim \left\{\sum_{j} 2^{-j}\left(\sum_{d<2j+5} 1\right)^2\right\}^{1/2}\\ &< +\infty.\end{aligned}$$

The second term is the sum of two norms,

$$\left\|\mathbf{1}_{D_2}\frac{w}{\langle\xi\rangle\langle\tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w}*\frac{g_{j_2}}{w}\right)\right\|_{L^2L^1} + \left\|\mathbf{1}_{D_2}\frac{w}{\langle\tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w}*\frac{g_{j_2}}{w}\right)\right\|_{L^2L^2}.$$

Note that the relation $\langle\tau-\xi^2\rangle\sim\langle\tau\rangle\ge\langle\xi^2\rangle$ holds in D_2, then the L^2L^2-norm is easily estimated by using (11),

$$\begin{aligned}\left\|\mathbf{1}_{D_2}\frac{w}{\langle\tau-\xi^2\rangle}\left(\frac{f_{j_1}}{w}*\frac{g_{j_2}}{w}\right)\right\|_{L^2L^2} &\lesssim \left\|\mathbf{1}_{D_2}\langle\tau-\xi^2\rangle^{-1}f_{j_1}*g_{j_2}\right\|_{L^2L^2}\\ &\lesssim \left\|\langle\tau\rangle^{-2/3}\langle\xi^2\rangle^{-1/3}\right\|_{L^2L^2}\|f_{j_1}*g_{j_2}\|_{L^\infty L^\infty}\\ &\lesssim \|f_{j_1}\|_{L^2L^2}\,\|g_{j_2}\|_{L^2L^2}\,.\end{aligned}$$

Estimate of the L^2L^1-norm is obtained from the following one-variable lemma.

Lemma Ex. *For all* $\varphi, \psi \in L^2(\mathbb{R})$*, we have*

$$\left\| \frac{w}{\langle \cdot \rangle} \left(\frac{\varphi}{w} * \frac{\psi}{w} \right) \right\|_{L^1(\mathbb{R})} \lesssim \|\varphi\|_{L^2(\mathbb{R})} \|\psi\|_{L^2(\mathbb{R})},$$

where $w(\tau) := [\min\{-1, \tau\}]^{10}$.

Using Hölder's inequality, Fubini's theorem and this lemma, we can actually evaluate the L^2L^1-norm:

$$\begin{aligned}
&\left\| \mathbf{1}_{D_2} \frac{w}{\langle \xi \rangle \langle \tau - \xi^2 \rangle} \left(\frac{f_{j_1}}{w} * \frac{g_{j_2}}{w} \right) \right\|_{L^2L^1} \sim \left\| \mathbf{1}_{D_2} \frac{w}{\langle \xi \rangle \langle \tau \rangle} \left(\frac{f_{j_1}}{w} * \frac{g_{j_2}}{w} \right) \right\|_{L^2L^1} \\
&\lesssim \left\| \frac{w}{\langle \tau \rangle} \left(\frac{f_{j_1}}{w} * \frac{g_{j_2}}{w} \right) \right\|_{L^\infty L^1} \\
&= \left\| \int_{\mathbb{R}} \left\| \frac{w}{\langle \cdot \rangle} \left(\frac{f_{j_1}(\cdot, \xi_1)}{w} *_\tau \frac{g_{j_2}(\cdot, \xi - \xi_1)}{w} \right) \right\|_{L^1_\tau} d\xi_1 \right\|_{L^\infty_\xi} \\
&\lesssim \left\| \|f_{j_1}(\cdot, \xi)\|_{L^2_\tau} *_\xi \|g_{j_2}(\cdot, \xi)\|_{L^2_\tau} \right\|_{L^\infty_\xi} \\
&\leq \|f_{j_1}\|_{L^2L^2} \|g_{j_2}\|_{L^2L^2}.
\end{aligned}$$

This concludes the proof of (16).

The proof of Lemma Ex becomes very simple if we make use of the boundedness of the *Hilbert transform.* The Hilbert transform $\mathcal{H}$ is a singular integral operator defined by

$$(\mathcal{H}f)(x) := \frac{1}{\pi} \text{ p.v.} \int_{\mathbb{R}} \frac{f(x-y)}{y} dy,$$

which is known as a bounded linear transform on $L^p(\mathbb{R})$, $1 < p < \infty$.

Proof of Lemma Ex. We may suppose both φ and ψ to be non-negative. Then,

$$\begin{aligned}
\left\| \frac{w}{\langle \cdot \rangle} \left(\frac{\varphi}{w} * \frac{\psi}{w} \right) \right\|_{L^1(\mathbb{R})} &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{w(\tau)\varphi(\tau_1)\psi(\tau_2)}{\langle \tau \rangle w(\tau_1) w(\tau_2)} d\tau_1 \, d\tau \quad (\tau_1 + \tau_2 = \tau) \\
&=: I.
\end{aligned}$$

We may also assume that φ, ψ are restricted either to $[0, \infty)$ or to $(-\infty, 0]$, and then it suffices to consider the following two cases:

(i) $\operatorname{supp} \varphi, \ \operatorname{supp} \psi \subset [0, \infty)$,
(ii) $\operatorname{supp} \varphi \subset [0, \infty)$ and $\operatorname{supp} \psi \subset (-\infty, 0]$,

since the other cases follow from a similar argument.

Case (i). We discard w's and use Fubini's theorem to have

$$\begin{aligned}
I &\lesssim \int_0^\infty \varphi(\tau_1) \int_{\tau_1}^\infty \frac{\psi(\tau - \tau_1)}{\langle \tau \rangle} d\tau \, d\tau_1 \\
&\leq \int_0^\infty \varphi(\tau_1) \int_{\tau_1}^\infty \frac{\psi_-(\tau_1 - \tau)}{\tau} d\tau \, d\tau_1 \\
&= \pi \int_0^\infty \varphi(\tau_1) \cdot (\mathcal{H}\psi_-)(\tau_1) d\tau_1,
\end{aligned}$$

where $\psi_-(\tau) := \psi(-\tau)$. Then Schwarz's inequality and L^2-boundedness of $\mathcal{H}$ imply the desired bound.

Case (ii). In this case, a simple calculation shows that

$$\frac{w(\tau)}{\langle\tau\rangle w(\tau_1)w(\tau_2)} \lesssim \frac{1}{\tau_1-\tau_2}, \quad \tau_1-\tau_2\geq 0$$

whenever $\tau_1 \in \operatorname{supp}\varphi$, $\tau_2 \in \operatorname{supp}\psi$ and $\tau=\tau_1+\tau_2$. Using Fubini's theorem again and changing the variable we have

$$\begin{aligned} I &\lesssim \int_0^\infty \varphi(\tau_1)\int_{-\infty}^0 \frac{\psi(\tau_2)}{\tau_1-\tau_2}d\tau_2\,d\tau_1 \\ &= \pi\int_0^\infty \varphi(\tau_1)\cdot(\mathcal{H}\psi)(\tau_1)d\tau_1, \end{aligned}$$

and obtain the same bound as in (i). □

4. **Proof for the second part.** Next, we consider Corollary 1.

In the proof of Corollary 1 in [2], the following estimate, which was proved in Proposition 5, was essentially used:

Assume that $\operatorname{supp} f_{j_1}\subset A_{j_1}\cap B_{d_1}$, $\operatorname{supp} g_{j_2}\subset A_{j_2}\cap B_{d_2}$, *and that there exists* $D\geq 0$ *such that* $|\xi_1-\xi_2|\geq D$ *whenever* $(\tau_1,\xi_1)\in\operatorname{supp} f_{j_1}$ *and* $(\tau_2,\xi_2)\in\operatorname{supp} g_{j_2}$. *Then,*

$$\|f*g\|_{L^2L^2}\lesssim 2^{(d_1+d_2)/2}\left(2^{d_1/2}+2^{d_2/2}+D\right)^{-1/2}\|f\|_{L^2L^2}\|g\|_{L^2L^2}. \tag{17}$$

The proof of Corollary 1 was then as follows:

From duality, Fubini's theorem and Schwarz's inequality, we have

$$\|f*g\|_{L^2L^2(\Omega\cap B_d)} = \sup_h\left|\int_{\mathbb{R}^2} h\cdot(f*g)\,d\tau\,d\xi\right|$$

$$\textit{(supremum over } h\in L^2L^2 \textit{ s.t. } \operatorname{supp} h\subset\Omega\cap B_d,\ \|h\|_{L^2L^2}\leq 1\textit{)}$$

$$\begin{aligned} &= \sup_h\left|\int_{\mathbb{R}^2} g\cdot(f_-*h)\,d\tau\,d\xi\right| \\ &\leq \|g\|_{L^2L^2}\cdot\sup_h\|f_-*h\|_{L^2L^2}, \end{aligned}$$

where $f_-(\tau,\xi) := f(-\tau,-\xi)$. *On the other hand, by decomposing* $f=\sum_{d_1} f_{d_1}$ *(in this proof* $f_{d_1} := f\cdot\boldsymbol{1}_{B_{d_1}}$*) and using* (17),

$$\begin{aligned} \|f_-*h\|_{L^2L^2} &\lesssim \sum_{d_1\geq 0} 2^{(d_1+d)/2}\left(2^{d_1/2}+2^{d/2}+D\right)^{-1/2}\|f_{d_1}\|_{L^2L^2}\|h\|_{L^2L^2} \quad (18)\\ &\leq 2^{d/2}\left(2^{d/2}+D\right)^{-1/2}\|h\|_{L^2L^2}\sum_{d_1\geq 0}2^{d_1/2}\|f_{d_1}\|_{L^2L^2} \\ &= 2^{d/2}\left(2^{d/2}+D\right)^{-1/2}\|h\|_{L^2L^2}\cdot 2^{j_1}\|f\|_{\hat{X}^{-1,1/2,1}}, \end{aligned}$$

and the claim follows. □

However, the estimate (18) is not clear; if we decompose f_- like $\sum(f_-)_{d_1}$, then $\|(f_-)_{d_1}\|_{L^2L^2}\neq\|f_{d_1}\|_{L^2L^2}$ because B_{d_1} is not symmetric with respect to the origin, and if we decompose $f_-=\sum(f_{d_1})_-$, then we cannot apply (17) because $(f_{d_1})_-$ is no longer supported on B_{d_1}.

In fact, there is an easy counterexample to Corollary 1. For $j \geq 0$, define P as the parallelogram with vertices

$$(\tau,\xi) = \left(2^{2j}, 2^j\right), \quad \left(2^{2j}+1, 2^j\right), \\ \left((2^j+1)^2, 2^j+1\right), \quad \left((2^j+1)^2-1, 2^j+1\right),$$

and P_0 as the parallelogram with the same shape and direction which is centered at the origin. Let $f := \mathbf{1}_P$ and $g := \mathbf{1}_{P_0}$. Then,

- supp f, supp $f * g \subset A_j$,
- $P \subset A_j \cap B_0$, $f * g \geq \frac{1}{4}\mathbf{1}_P$,
- $|\xi + \xi_1| \sim 2^j$ if $(\tau,\xi) \in$ supp $f * g$ and $(\tau_1,\xi_1) \in$ supp f,
- $\|f\|_{\hat{X}^{-1,1/2,1}} \sim 2^{-j}$, $\|g\|_{L^2L^2} \sim 1$ and $\|f * g\|_{L^2L^2(A_j \cap B_0)} \sim 1$.

So Corollary 1 with $d = 0$, $\Omega = A_j$ and $D = 2^j$ implies $1 \lesssim 2^{-j/2}$, which is a contradiction for large j's.

It is possible that Corollary 1 holds without the term of D, but this term is important for the proof of the bilinear estimate.

We now prove a new proposition instead of Corollary 1.

Proposition Ex. *Suppose that* supp $f \subset A_{j_1}$, supp $g \subset A_{j_2}$ *for some* $j_1 \geq 0$ *and* $j_2 > 0$. *Then,*

$$2^{-d/2}\|f * g\|_{L^2L^2(B_d)} \lesssim 2^{j_1}2^{-j_2/2}\|f\|_{\hat{X}^{-1,1/2,1}}\|g\|_{L^2L^2}$$

for any $d \geq 0$.

Proof. We may assume f and g to be non-negative without loss of generality. By the argument in the proof of Corollary 1, it suffices to show that

$$\|(f_{d_1})_- * h\|_{L^2L^2(A_{j_2})} \lesssim 2^{(d_1+d)/2}2^{-j_2/2}\|f_{d_1}\|_{L^2L^2}\|h\|_{L^2L^2}$$

for $d_1 \geq 0$ and any non-negative test function $h \in L^2L^2$ restricted to B_d. We use Schwarz's inequality and Fubini's theorem to have

$$\begin{aligned}
&\|(f_{d_1})_- * h\|_{L^2L^2(A_{j_2})} \\
&\quad = \left\| \int_{\mathbb{R}^2} f_{d_1}(\tau_1,\xi_1)h(\tau_1+\tau_2,\xi_1+\xi_2)\,d\tau_1 d\xi_1 \right\|_{L^2_{\xi_2}L^2_{\tau_2}(A_{j_2})} \\
&\quad \leq \left\| m_{d_1}(\tau_2,\xi_2)^{1/2}\left(\int_{\mathbb{R}^2} f_{d_1}(\tau_1,\xi_1)^2 h(\tau_1+\tau_2,\xi_1+\xi_2)^2\,d\tau_1 d\xi_1\right)^{1/2} \right\|_{L^2_{\xi_2}L^2_{\tau_2}(A_{j_2})} \\
&\quad \leq \left[\sup_{(\tau_2,\xi_2)\in A_{j_2}} m_{d_1}(\tau_2,\xi_2)\right]^{1/2} \|f_{d_1}\|_{L^2L^2}\|h\|_{L^2L^2},
\end{aligned}$$

where $m_{d_1}(\tau_2,\xi_2)$ is the measure of the set

$$\{(\tau_1,\xi_1) \in A_{j_1} \cap B_{d_1} \,:\, (\tau_1+\tau_2,\xi_1+\xi_2) \in B_d\}.$$

Thus it suffices to show

$$m_{d_1}(\tau_2,\xi_2) \lesssim 2^{d_1+d-j_2} \tag{19}$$

for $(\tau_2,\xi_2) \in A_{j_2}$.

Assume $(\tau_1,\xi_1) \in A_{j_1} \cap B_{d_1}$, $(\tau_2,\xi_2) \in A_{j_2}$ and $(\tau_1+\tau_2,\xi_1+\xi_2) \in B_d$. We have

$$\begin{aligned}
\left|\xi_1 + \frac{1}{2}\left(\xi_2 - \frac{\tau_2}{\xi_2}\right)\right| &= \frac{1}{2|\xi_2|}\left|(\tau_1-\xi_1^2) - \left\{\tau_1+\tau_2-(\xi_1+\xi_2)^2\right\}\right| \\
&\lesssim 2^{-j_2}\left(2^{d_1}+2^d\right),
\end{aligned}$$

which means that the variation of ξ_1 is estimated by $2^{-j_2}\left(2^{d_1}+2^{d}\right)$ for fixed (ξ_2,τ_2). (We have assumed j_2 to be nonzero, so $|\xi_2|\geq 1$ and $\frac{1}{|\xi_2|}\sim 2^{-j_2}$.)

If we also fix ξ_1, then the estimates

$$|\tau_1-\xi_1^2|\lesssim 2^{d_1}\ ,\quad |\tau_1+\tau_2-(\xi_1+\xi_2)^2|\lesssim 2^{d}$$

imply that the variation of τ_1 is bounded by $\min\left\{2^{d_1},2^{d}\right\}$. We therefore obtain (19). □

In [2], Corollary 1 was used only in the proof of (9). More precisely, the L.H.S. of (9) was divided into three pieces,

$$\begin{aligned}\left\|\mathbf{1}_{A_j}\langle\tau-\xi^2\rangle^{-1}f_{j_1}*g_{j_2}\right\|_Z \leq{}& \left\|\mathbf{1}_{A_j}\langle\tau-\xi^2\rangle^{-1}f_{j_1,\geq j+j_2-20}*g_{j_2}\right\|_{\hat{X}^{-1,1/2,1}}\\ &+\left\|\mathbf{1}_{A_j}\langle\tau-\xi^2\rangle^{-1}f_{j_1,<j+j_2-20}*g_{j_2,\geq j+j_2-20}\right\|_{\hat{X}^{-1,1/2,1}}\\ &+\left\|\mathbf{1}_{A_j}\langle\tau-\xi^2\rangle^{-1}f_{j_1,<j+j_2-20}*g_{j_2,<j+j_2-20}\right\|_{\hat{X}^{-1,1/2,1}}\,,\end{aligned}$$

and Corollary 1 was used to estimate the first and the second terms.

If we try to prove (9) by using Proposition Ex, another (finer) division will be needed. Recall that $|j_1-j|\leq 10$ and $j_2\leq j_1+11$. (9) is easy to verify in the case $j_1=0$ or $j_2=0$, so we assume $j_1>0$ and $j_2>0$. Then the assumption $|j_1-j|\leq 10$ and the identity

$$(\tau-\xi^2)-(\tau_1-\xi_1^2)-(\tau_2-\xi_2^2)=-2\xi_1\xi_2$$

with the convention (8) imply the *resonance estimate*

$$\begin{aligned}\max\left\{\langle\tau-\xi^2\rangle,\langle\tau_1-\xi_1^2\rangle,\langle\tau_2-\xi_2^2\rangle\right\}&\gtrsim 2^{j+j_2},\\ &(\text{say }\ \geq 2^{j+j_2-20}.)\end{aligned}\tag{20}$$

We first consider the case $j_2\geq\varepsilon j_1$, $0<\varepsilon\ll 1$. Noting the estimate (20), we divide the L.H.S. of (9) into four,

$$\begin{aligned}&\left\|\mathbf{1}_{A_j}\langle\tau-\xi^2\rangle^{-1}f_{j_1}*g_{j_2}\right\|_Z\\ &\leq\left\|\mathbf{1}_{A_j\cap B_{\geq j+j_2-20}}\langle\tau-\xi^2\rangle^{-1}f_{j_1,\geq j+j_2-20}*g_{j_2}\right\|_Y\\ &\quad+\left\|\mathbf{1}_{A_j\cap B_{\geq j+j_2-20}}\langle\tau-\xi^2\rangle^{-1}f_{j_1,<j+j_2-20}*g_{j_2}\right\|_{\hat{X}^{-1,1/2,1}}\\ &\quad+\left\|\mathbf{1}_{A_j\cap B_{<j+j_2-20}}\langle\tau-\xi^2\rangle^{-1}f_{j_1,\geq j+j_2-20}*g_{j_2}\right\|_{\hat{X}^{-1,1/2,1}}\\ &\quad+\left\|\mathbf{1}_{A_j\cap B_{<j+j_2-20}}\langle\tau-\xi^2\rangle^{-1}f_{j_1,<j+j_2-20}*g_{j_2,\geq j+j_2-20}\right\|_{\hat{X}^{-1,1/2,1}}\\ &=:\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.\end{aligned}$$

Estimate of I. From the definition of Y we have

$$\mathrm{I}\lesssim 2^{-j-j_2}\left(2^{-j}\left\|f_{j_1,\geq j+j_2-20}*g_{j_2}\right\|_{L^2L^1}+\left\|f_{j_1,\geq j+j_2-20}*g_{j_2}\right\|_{L^2L^2}\right).$$

We use Young's inequality, (12) and (14) to obtain

$$\begin{aligned}\left\|f_{j_1}*g_{j_2}\right\|_{L^2L^1}&\leq\left\|f_{j_1}\right\|_{L^2L^1}\left\|g_{j_2}\right\|_{L^1L^1}\\ &\lesssim 2^{j_1}2^{3j_2/2}\left\|f_{j_1}\right\|_Z\left\|g_{j_2}\right\|_Z\,,\end{aligned}$$

and also use (13) to obtain

$$\begin{aligned}\left\|f_{j_1,\geq j+j_2-20}*g_{j_2}\right\|_{L^2L^2}&\leq\left\|f_{j_1,\geq j+j_2-20}\right\|_{L^2L^2}\left\|g_{j_2}\right\|_{L^1L^1}\\ &\lesssim 2^{j_1-(j+j_2)/2}2^{3j_2/2}\left\|f_{j_1}\right\|_Z\left\|g_{j_2}\right\|_Z\,.\end{aligned}$$

Therefore,

$$\mathrm{I}\lesssim 2^{-j_2/2}\left\|f_{j_1}\right\|_Z\left\|g_{j_2}\right\|_Z\,.$$

Estimate of II. From Lemma 1,

$$\begin{aligned}\mathrm{II} &\lesssim 2^{-(j+j_2)/2} \left\| \mathbf{1}_{A_j} \langle \tau - \xi^2 \rangle^{-1} f_{j_1} * g_{j_2} \right\|_{\hat{X}^{-1,1}} \\ &\sim 2^{-(j+j_2)/2} 2^{-j} \left\| f_{j_1} * g_{j_2} \right\|_{L^2 L^2} .\end{aligned}$$

By Lemma 2, we may measure f_{j_1} in $\hat{X}^{-1,1/2,1}$ instead of in Z. If we use $\hat{X}^{-1,1/2,1}$ to estimate g_{j_2}, by Proposition 5 with $D = 0$ we have

$$\left\| f_{j_1} * g_{j_2} \right\|_{L^2 L^2} \lesssim 2^{j_1 + j_2} \left\| f_{j_1} \right\|_{\hat{X}^{-1,1/2,1}} \left\| g_{j_2} \right\|_{\hat{X}^{-1,1/2,1}} .$$

If we use Y, then Young's inequality, (12) and (15) imply that

$$\begin{aligned}\left\| f_{j_1} * g_{j_2} \right\|_{L^2 L^2} &\lesssim \left\| f_{j_1} \right\|_{L^2 L^1} \left\| g_{j_2} \right\|_{L^1 L^2} \\ &\lesssim 2^{j_1} 2^{j_2/2} \left\| f_{j_1} \right\|_{\hat{X}^{-1,1/2,1}} \left\| g_{j_2} \right\|_{Y} .\end{aligned}$$

Thus,

$$\mathrm{II} \lesssim 2^{-(j-j_2)/2} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Z .$$

Estimate of III. From the definition of the $\hat{X}^{-1,1/2,1}$-norm,

$$\mathrm{III} \sim 2^{-j} \sum_{d < j + j_2 - 20} 2^{-d/2} \left\| f_{j_1, \geq j + j_2 - 20} * g_{j_2} \right\|_{L^2 L^2 (A_j \cap B_d)} .$$

We use Proposition Ex and (13) to obtain

$$\begin{aligned}&2^{-d/2} \left\| f_{j_1, \geq j + j_2 - 20} * g_{j_2} \right\|_{L^2 L^2 (A_j \cap B_d)} \\ &\quad \lesssim 2^{j_2} 2^{-j_1/2} \left\| f_{j_1, \geq j + j_2 - 20} \right\|_{L^2 L^2} \left\| g_{j_2} \right\|_{\hat{X}^{-1,1/2,1}} \\ &\quad \lesssim 2^{j_2} 2^{-j_1/2} 2^{j_1 - (j+j_2)/2} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_{\hat{X}^{-1,1/2,1}} ,\end{aligned}$$

then

$$\begin{aligned}\mathrm{III} &\lesssim 2^{-j} 2^{j_2/2} \left(\sum_{d < j + j_2 - 20} 1 \right) \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_{\hat{X}^{-1,1/2,1}} \\ &\lesssim 2^{-(j-j_2)/2} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_{\hat{X}^{-1,1/2,1}} .\end{aligned}$$

On the other hand, we use Lemma 1, Hölder's inequality and Young's inequality to have

$$\begin{aligned}\mathrm{III} &\lesssim \left\| \mathbf{1}_{A_j} \langle \tau - \xi^2 \rangle^{-1} f_{j_1, \geq j + j_2 - 20} * g_{j_2} \right\|_{\hat{X}^{-1,1/2+\delta}} \qquad (0 < \delta \ll 1) \\ &\sim 2^{-j} \left\| \langle \tau - \xi^2 \rangle^{-1/2+\delta} f_{j_1, \geq j + j_2 - 20} * g_{j_2} \right\|_{L^2 L^2} \\ &\lesssim 2^{-j} \left\| \langle \tau - \xi^2 \rangle^{-1/2+\delta} \right\|_{L^\infty L^3} \left\| f_{j_1, \geq j + j_2 - 20} * g_{j_2} \right\|_{L^2 L^6} \\ &\lesssim 2^{-j} \left\| f_{j_1, \geq j + j_2 - 20} \right\|_{L^2 L^2} \left\| g_{j_2} \right\|_{L^1 L^{3/2}} ,\end{aligned}$$

and using (13)–(15) we obtain

$$\begin{aligned}\mathrm{III} &\lesssim 2^{-j} \left\| f_{j_1, \geq j + j_2 - 20} \right\|_{L^2 L^2} \left\| g_{j_2} \right\|_{L^1 L^1}^{1/3} \left\| g_{j_2} \right\|_{L^1 L^2}^{2/3} \\ &\lesssim 2^{-j} 2^{j_1 - (j+j_2)/2} 2^{j_2/2} 2^{j_2/3} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Y \\ &\lesssim 2^{-(j-j_2)/3} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Y .\end{aligned}$$

Therefore we obtain

$$\mathrm{III} \lesssim 2^{-(j-j_2)/3} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Z .$$

Estimate of IV. We use Proposition Ex, Lemma 2 and (13) to have

$$\begin{aligned}
\mathrm{IV} &\sim 2^{-j} \sum_{d<j+j_2-20} 2^{-d/2} \left\| f_{j_1,<j+j_2-20} * g_{j_2,\geq j+j_2-20} \right\|_{L^2L^2(A_j\cap B_d)} \\
&\lesssim 2^{-j} \left(\sum_{d<j+j_2-20} 1 \right) 2^{j_1} 2^{-j_2/2} \left\| f_{j_1} \right\|_{\hat{X}^{-1,1/2,1}} \left\| g_{j_2,\geq j+j_2-20} \right\|_{L^2L^2} \\
&\lesssim j\, 2^{-j_2/2} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Z .
\end{aligned}$$

We finally use the assumption $j_2 \geq \varepsilon j_1$ to obtain

$$\mathrm{IV} \lesssim 2^{-j_2/4} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Z .$$

Next, we consider the case where $j_2 < \varepsilon j_1$. In this case, we divide the L.H.S. of (9) in a different way, based on the estimate (20),

$$\begin{aligned}
&\left\| \mathbf{1}_{A_j} \langle \tau - \xi^2 \rangle^{-1} f_{j_1} * g_{j_2} \right\|_Z \\
&\quad \leq \left\| \mathbf{1}_{A_j \cap B_{\geq 2j_2-40}} \langle \tau - \xi^2 \rangle^{-1} f_{j_1,\geq 2j} * g_{j_2} \right\|_Y \\
&\qquad + \left\| \mathbf{1}_{A_j \cap B_{\geq 2j_2-40}} \langle \tau - \xi^2 \rangle^{-1} f_{j_1,<2j} * g_{j_2} \right\|_{\hat{X}^{-1,1/2,1}} \\
&\qquad + \left\| \mathbf{1}_{A_j \cap B_{<2j_2-40}} \langle \tau - \xi^2 \rangle^{-1} f_{j_1,\geq 2j} * g_{j_2} \right\|_{\hat{X}^{-1,1/2,1}} \\
&\qquad + \left\| \mathbf{1}_{A_j \cap B_{<2j_2-40}} \langle \tau - \xi^2 \rangle^{-1} f_{j_1,<2j} * g_{j_2,\geq j+j_2-20} \right\|_{\hat{X}^{-1,1/2,1}} \\
&\qquad + \left\| \mathbf{1}_{A_j \cap B_{<2j_2-40}} \langle \tau - \xi^2 \rangle^{-1} f_{j_1,[j+j_2-20,2j)} * g_{j_2,<j+j_2-20} \right\|_{\hat{X}^{-1,1/2,1}} \\
&\quad =: \mathrm{I}' + \mathrm{II}' + \mathrm{III}' + \mathrm{IV}' + \mathrm{V}'.
\end{aligned}$$

I'–IV' can be estimated in a similar way to the estimate of I–IV. Note that the assumption $j_2 < \varepsilon j_1$ allows us to use Proposition 5 with $D \sim 2^{j_1}$ in estimating II'.

Estimate of V′. We use Lemma 1, Young's inequality and (13), (14) to have

$$\begin{aligned}
\mathrm{V}' &\lesssim \left\| \mathbf{1}_{A_j} \langle \tau - \xi^2 \rangle^{-1} f_{j_1,[j+j_2-20,2j)} * g_{j_2} \right\|_{\hat{X}^{-1,1}} \\
&\lesssim 2^{-j} \left\| f_{j_1,[j+j_2-20,2j)} \right\|_{L^2L^2} \left\| g_{j_2} \right\|_{L^1L^1} \\
&\lesssim 2^{-j} 2^{j_1 - (j+j_2)/2} 2^{3j_2/2} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Z \\
&= 2^{-j/2} 2^{j_2} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Z ,
\end{aligned}$$

and using $j_2 < \varepsilon j_1$ we obtain

$$\mathrm{V}' \lesssim 2^{-j_2} \left\| f_{j_1} \right\|_Z \left\| g_{j_2} \right\|_Z .$$

This concludes the proof of (9).

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