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ON BANACH ALGEBRAS OF SOME MATRIX CLASSES

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Abstract. In this article we characterize some matrix classes involving some difference sequence spaces and the spaces c and ℓ_∞. We show that these matrix classes can be made Banach algebras and prove that these matrix classes are semisimple. Further we investigate the topologically and algebraically equivalent spaces. This article also introduces the concept of application of generalized difference operator to infinite matrices. These investigations generalize several notions associated with matrix transformations.

1. **Introduction.** Let w denote the space of all real or complex sequences. By c, c_0 and ℓ_∞, we denote the Banach spaces of convergent, null and bounded sequences $x = (x_k)$, respectively normed by

$$\|x\| = \sup_k |x_k| \tag{1}$$

Let E and F be two sequence spaces and $A = (a_{nk})$ be an infinite matrix of real or complex numbers a_{nk}, where $n, k \in N$. Then, we say that A defines a matrix mapping from E into F, and denote it by writing $A : E \longrightarrow F$ if for every sequence $x = (x_k) \in E$ the sequence $Ax = \{(Ax)_n\}$, the $A-$transform of x, is in F, where

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k, \ (n \in N) \tag{2}$$

We denote by (E, F) the class of all matrices A such that $A : E \longrightarrow F$. Thus, $A \in (E, F)$ if and only if the series on the right hand side of (2) converges for each $n \in N$ and every $x \in E$, and we have $Ax = \{(Ax)_n\}_{n\in N} \in F$ for all $x \in E$. A sequence x is said to be $A-$summable to l if Ax converges to l which is called the $A-$limit of x.

Interest in general matrix transformation theory was, to some extent, stimulated by special results in summability theory which were obtain by Cesàro, Borel and others, at the turn of the 20^{th} century. It was however the celebrated German mathematician O. Toeplitz who, in 1911, brought the methods of linear space theory to bear on problems connected with matrix transformations on sequence spaces. Toeplitz characterized all those infinite matrices $A = (a_{nk})$, $n, k \in N$, which map the space of convergent sequences into itself, leaving the limit of each convergent sequence invariant.

In mathematics, Banach spaces (pronounced as 'banax') are one of the central objects of study in functional analysis. Many of the infinite-dimensional function

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spaces studied in analysis are Banach spaces, including spaces of continuous functions (continuous functions on a compact Hausdorff space), spaces of Lebesgue integrable functions known as L_p spaces, and spaces of holomorphic functions known as Hardy spaces. They are the most commonly used topological vector spaces, and their topology comes from a norm.

They are named after the Polish mathematician Stefan Banach, who introduced them in 1920-1922 along with Hans Hahn and Eduard Helly. Some famous Banach spaces in other areas of analysis are Hardy spaces, space BMO of functions of bounded mean oscillation, space of functions of bounded variation, Sobolev spaces, Birnbaum-Orlicz spaces, Hölder spaces and Lorentz spaces.

Banach spaces are defined as complete normed vector spaces. This means that a Banach space is a vector space V over the real or complex numbers with a norm $\|.\|$ such that every Cauchy sequence (with respect to the metric $d(x, y) = \|x - y\|$) in V has a limit in V.

In functional analysis, a Banach algebra, named after Stefan Banach, is an associative algebra A over the real or complex numbers which at the same time is also a Banach space. The algebra multiplication and the Banach space norm are required to be related by the following inequality:

$$\|xy\| \leq \|x\|\|y\|, \text{ for all } x, y \in A.$$

(i.e., the norm of the product is less than or equal to the product of the norms.) This ensures that the multiplication operation is continuous.

If in the above we relax Banach space to normed space the analogous structure is called a normed algebra.

An isometry, isometric isomorphism or congruence mapping is a distance-preserving map between metric spaces or normed spaces. Geometric figures which can be related by an isometry are called congruent.

Isometries are often used in constructions where one space is embedded in another space. For instance, the completion of a metric space M involves an isometry from M into M', a quotient set of the space of Cauchy sequences on M. The original space M is thus isometrically isomorphic to a subspace of a complete metric space, and it is usually identified with this subspace. Other embedding constructions show that every metric space is isometrically isomorphic to a closed subset of some normed vector space and that every complete metric space is isometrically isomorphic to a closed subset of some Banach space.

The notion of difference sequence space was introduced by Kizmaz [7], who studied the difference sequence spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [5] by introducing the spaces $\ell_\infty(\Delta^s)$, $c(\Delta^s)$ and $c_0(\Delta^s)$. Another type of generalization of the difference sequence spaces is due to Tripathy and Esi [10], who studied the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$. Tripathy, Esi and Tripathy [12] generalized the above notions and unified these as follows:

Let m, s be non-negative integers, then for Z a given sequence space we have

$$Z(\Delta_m^s) = \{x = (x_k) \in w : (\Delta_m^s x_k) \in Z\},$$

where $\Delta_m^s x = (\Delta_m^s x_k) = (\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation

$$\Delta_m^s x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{k+mv}.$$

Tripathy, Esi and Sarma [6] showed that $c_0(\Delta_m^s)$, $c(\Delta_m^s)$ and $\ell_\infty(\Delta_m^s)$ are Banach sapces normed by

$$\|x\| = \sum_{k=1}^{ms} |x_k| + \sup_k |\Delta_m^s x_k| \tag{3}$$

Taking $m = 1$, we get the spaces $\ell_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_0(\Delta^n)$ studied by Et and Colak [5]. Taking $s = 1$, we get the spaces $\ell_\infty(\Delta_m)$, $c(\Delta_m)$ and $c_0(\Delta_m)$ studied by Tripathy and Esi [10]. Taking $m = s = 1$, we get the spaces $\ell_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kizmaz [7].

Let m, s be non-negative integers, then for Z a given sequence space Dutta [1] introduced

$$Z(\Delta_m^{(s)}) = \{x = (x_k) \in w : (\Delta_m^{(s)} x_k) \in Z\},$$

where $\Delta_m^{(s)} x = (\Delta_m^{(s)} x_k) = (\Delta_m^{(s-1)} x_k - \Delta_m^{(s-1)} x_{k-m})$ and $\Delta_m^{(0)} x_k = x_k$ for all $k \in N$, which is equivalent to the following binomial representation

$$\Delta_m^{(s)} x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{k-mv}.$$

It is important here to note that we take $x_{k-mv} = 0$ for non-positive values of $k - mv$.

It can be shown that the spaces $c_0(\Delta_m^{(s)})$, $c(\Delta_m^{(s)})$ and $\ell_\infty(\Delta_m^{(s)})$ are Banach spaces normed by

$$\|x\| = \sup_k |\Delta_m^{(s)} x_k| \tag{4}$$

It is obvious that $(x_k) \in Z(\Delta_m^{(s)})$ if and only if $(x_k) \in Z(\Delta_m^s)$. But if we compare the norms (3) and (4) with norm (1), (4) looks quite natural as norm on a generalized space of c, c_0 and ℓ_∞. Keeping this in mind this new operator $\Delta_m^{(s)}$ was introduced. Some more usefulness of this operator will be visible in the next section.

Recently Dutta and Reddy [4] used the difference operator $\Delta_m^{(s)}$ to construct some sequence spaces and studied these spaces by defining non-standard n-norm $(n \geq 2)$.

Dutta [3] used the difference operators Δ_r and $\Delta_{(r)}$ to infinite matrix of non-negative real numbers to construct the sequence spaces $(\hat{A}, p, \Delta_{(r)})_0$, $(\hat{A}, p, \Delta_r)_0$, $(\hat{A}, p, \Delta_{(r)})$, $(\hat{A}, p, \Delta_r)$, $(\hat{A}, p, \Delta_{(r)})_\infty$ and $(\hat{A}, p, \Delta_r)_\infty$ respectively. In the same paper it was shown that if we restrict the class of matrices to one which include the infinite matrices $A = (a_{nk})$ of non-negative real numbers with $\alpha_{i1} = \Delta_{(r)} a_{i1} \neq 0$ and $\beta_{ij} = |\Delta_{(r)} a_{ij} - \Delta_{(r)} a_{i,j-1}| = 0$, for every i, j, then the spaces become complete paranormed spaces.

2. **Main Results.**

In this section we characterize the matrix classes $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$, $(\ell_\infty, \ell_\infty(\Delta_m^s))$, $(c, c(\Delta_m^{(s)}))$ and $(c, c(\Delta_m^s))$. We show that these matrix classes can be made Banach algebras with respect to the matrix product and under a suitable norm and these spaces are semisimple. Further we compute topologically and algebraically equivalent spaces.

Theorem 1. *$A \in (\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ if and only if $\sup_j \sum_{k=1}^{\infty} |\Delta_m^{(s)} a_{jk}| < \infty$, where $\Delta_m^{(s)} a_{jk} = \sum_{v=0}^{s} (-1)^v \binom{s}{v} a_{j-mv,k}$ and we take $a_{j-mv,k} = 0$ for non-positive values*

of $j - rv$.
(e.g., $\Delta_3^{(2)} a_{11} = a_{11} - 2a_{-2,1} + a_{-5,1} = a_{11}$, $\Delta_3^{(2)} a_{71} = a_{71} - 2a_{41} + a_{11}$ *etc.,)*

Proof. Suppose $A \in (\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$. We put $B_j(x) = \Delta_m^{(s)}(Ax)_j$, where

$$\Delta_m^{(s)}(Ax)_j = \Delta_m^{(s-1)}(Ax)_j - \Delta_r^{(s-1)}(Ax)_{j-m} = \sum_{k=1}^{\infty} \Delta_m^{(s)} a_{jk} x_k, \ j = 1, 2 \cdots$$

Then we observe that (B_j) is a sequence of bounded linear operators on ℓ_∞ such that $\sup_j |B_j(x)| < \infty$. Now the result follows from an application of uniform boundedness principle. The sufficiency part is easy and so omitted. □

Theorem 2. $A \in (\ell_\infty, \ell_\infty(\Delta_m^s))$ *if and only if* $\sup_j \sum_{k=1}^{\infty} |\Delta_m^s a_{jk}| < \infty$.

Proof. Proof follows by similar arguments as applied to prove above Theorem. □

Theorem 3. $A \in (c, c(\Delta_m^{(s)}))$ *if and only if (i)* $\sup_j \sum_{k=1}^{\infty} |\Delta_m^{(s)} a_{jk}| < \infty$,
(ii) $\lim_{j\to\infty} \Delta_m^{(s)} a_{jk}$ *exists,* $k = 1, 2, \cdots$ *and*
(iii) $\lim_{j\to\infty} \sum_{k=1}^{\infty} \Delta_m^{(s)} a_{jk}$ *exists.*

Proof. Suppose $A \in (c, c(\Delta_m^{(s)}))$. The proof of necessity of (i) is same as that Theorem 1. Considering the convergent sequence $e_k = (0, 0, \cdots, 1, 0, \cdots)$, $k = 1, 2, \cdots$, where 1 is in the k^{th} position and $e = (1, 1, \cdots)$, the necessities (ii) and (iii) hold. Conversely let (x_k) converges to l and the condition (i), (ii) and (iii) hold. Then the sufficiency follows from the following equality

$$\sum_{k=1}^{\infty} \Delta_m^{(s)} a_{jk} x_k = \sum_{k=1}^{\infty} \Delta_m^{(s)} a_{jk}(x_k - l) + l \sum_{k=1}^{\infty} \Delta_m^{(s)} a_{jk}.$$

□

Theorem 4. $A \in (c, c(\Delta_m^s))$ *if and only if (i)* $\sup_j \sum_{k=1}^{\infty} |\Delta_m^s a_{jk}| < \infty$,
(ii) $\lim_{j\to\infty} \Delta_m^s a_{jk}$ *exists,* $k = 1, 2, \cdots$ *and*
(iii) $\lim_{j\to\infty} \sum_{k=1}^{\infty} \Delta_m^s a_{jk}$ *exists.*

Proof. Proof is similar to that of Theorem 3. □

Remark 1. Taking $s = 0$, in the above Theorems we get the famous matrix classes $(\ell_\infty, \ell_\infty)$ and (c, c).

Proposition 1. *(i)* $(\ell_\infty, \ell_\infty(\Delta_m^{(p)})) \subset (\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$, $p = 0, 1, \cdots, s-1$.
(ii) $(c, c(\Delta_m^{(p)})) \subset (c, c(\Delta_m^{(s)}))$, $p = 0, 1, \cdots, s-1$.

Proof. Proof follows from the fact that$Z(\Delta_m^{(p)}) \subset Z((\Delta_m^{(s)})$, $p = 0, 1, \cdots, s-1$ and $Z = \ell_\infty$, c. □

Remark 2. It is obvious that similar results that of above Proposition hold for the classes $(\ell_\infty, \ell_\infty(\Delta_m^s))$ and $(c, c(\Delta_m^s))$. In particular $(\ell_\infty, \ell_\infty)$ is a subspace of both $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$ and (c, c) is a subspace of both $(c, c(\Delta_m^{(s)}))$ and $(c, c(\Delta_m^s))$.

Our next aim is to show that $(\ell_\infty, \ell_\infty(\Delta_m^s))$, $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$, $(c, c(\Delta_m^{(s)}))$ and $(c, c(\Delta_m^s))$ are Banach algebras.

If $A = (a_{jk})$ and $B = (b_{jk})$ are two infinite matrices, the matrix product is defined by

$$(AB)_{jk} = \sum_{i=1}^{\infty} a_{ji} b_{ik}$$

Theorem 5. *$(c, c(\Delta_m^{(s)}))$ and $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ are Banach algebras with respect to the matrix product and the norm defined by*

$$\|A\| = \sup_j \sum_{k=1}^{\infty} |\Delta_m^{(s)} a_{jk}| \tag{5}$$

Proof. Proof is easy and so omitted. (see for instance [1]) □

Theorem 6. *$(c, c(\Delta_m^s))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$ are Banach algebras with respect to the matrix product and the norm defined by*

$$\|A\|' = \sum_{j=1}^{ms} |a_{jk}| + \sup_j \sum_{k=1}^{\infty} |\Delta_m^s a_{jk}|, \ k = 1, 2, \cdots \tag{6}$$

Proof. Proof is similar to that of above theorem. (see for instance [1]) □

Remark 3. It is obvious that the norms $\|.\|$ and $\|.\|'$ defined by (5) and(6) are equivalent.

In the next Theorem we show that $(c, c(\Delta_m^{(s)}))$, $(c, c(\Delta_m^s))$, $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$ are semisimple. Recall that a Banach algebra is said to be semisimple if the radical contains only zero.

Theorem 7. *$(c, c(\Delta_m^{(s)}))$, $(c, c(\Delta_m^s))$, $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$ are semisimple.*

Proof. We give the Proof for $(c, c(\Delta_m^{(s)}))$ only. For other cases it will follow on applying similar arguments. Let $A \in (c, c(\Delta_m^{(s)}))$, $A \neq 0$. Also let $x \in c$ and $Ax \neq 0$. Now we define $B \in (c, c(\Delta_m^{(s)}))$ in such a way that B consists entirely of zeros except for a single column and $BAx = x$. Then $I - BA$ maps x into 0, hence is not one-one and inverse does not exist. So, we can conclude that A cannot be in the radical of $(c, c(\Delta_m^{(s)}))$. This completes the proof. □

Now our aim is to investigate the algebraically equivalent spaces of $(c, c(\Delta_m^{(s)}))$, $(c, c(\Delta_m^s))$, $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$.

Proposition 2. *(i) The spaces $(c, c(\Delta_m^{(s)}))$ and $(c, c(\Delta_m^s))$, are isometrically isomorphic to the space (c, c).*
(ii) The spaces $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$ are isometrically isomorphic to the space $(\ell_\infty, \ell_\infty)$.

Proof. (i) It is obvious that $A \in (c, c(\Delta_m^s))$ if and only if $A \in (c, c(\Delta_m^{(s)}))$. Hence we can define the following mapping for $z = c(\Delta_m^{(s)})$, $c(\Delta_m^s)$,
$T : (c, z) \longrightarrow (c, c)$ defined by

$$TA = (\Delta_m^{(s)} a_{jk}) \ for \ every \ A \ in \ (c, z) \tag{7}$$

Then clearly A is linear, one-one and onto. Also

$$\|A\| = \sup_j \sum_{k=1}^{\infty} |\Delta_m^{(s)} a_{jk}| = \|TA\|$$

Thus T is an isometry.
(ii) Proof is similar with that of part (i). □

Next our aim is to find the topologically equivalent spaces of $(c, c(\Delta_m^{(s)}))$, $(c, c(\Delta_m^s))$, $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$. Consequently we get an interesting result that $(c, c(\Delta_m^s))$ and $(\ell_\infty, \ell_\infty(\Delta_m^s))$ have subspaces which are topologically equivalent to (c, c) and $(\ell_\infty, \ell_\infty)$ respectively.

Proposition 3. *(i) The space $(c, c(\Delta_m^{(s)}))$ is topologically isomorphic to the space (c, c).*
(ii) The space $S(c, c(\Delta_m^s))$ is topologically isometric to the space (c, c), where $S(c, c(\Delta_m^s))$ is a subspace of $(c, c(\Delta_m^s))$ defined by

$S(c, c(\Delta_m^s)) = \{A = (a_{jk}) : A \in (c, c(\Delta_m^s)),\ a_{jk} = 0,\ j = 1, 2, \cdots, ms \text{ and } k = 1, 2, \cdots\}$

and normed by

$$\|A\|' = \sup_j \sum_{k=1}^{\infty} |\Delta_m^s a_{jk}|$$

(iii) The space $(\ell_\infty, \ell_\infty(\Delta_m^{(s)}))$ is topologically isomorphic to the space $(\ell_\infty, \ell_\infty)$,
(iv) The space $S(\ell_\infty, \ell_\infty(\Delta_m^s))$ is topologically isomorphic to the space $(\ell_\infty, \ell_\infty)$, where $S(\ell_\infty, \ell_\infty(\Delta_m^s))$ is a subspace of $(\ell_\infty, \ell_\infty(\Delta_m^s))$ defined by $S(\ell_\infty, \ell_\infty(\Delta_m^s)) = \{A = (a_{jk}) : A \in (\ell_\infty, \ell_\infty(\Delta_m^s)),\ a_{jk} = 0,\ j = 1, 2, \cdots, ms \text{ and } k = 1, 2, \cdots\}$ and normed by

$$\|A\|' = \sup_j \sum_{k=1}^{\infty} |\Delta_m^s a_{jk}|$$

Proof. Here we give the proof of part (i) only. Proof of other parts follows on applying similar arguments. If we define a mapping from $(c, c(\Delta_m^{(s)}))$ into (c, c) exactly similar to (7), then clearly this mapping will be a linear homeomorphism. □

3. **Conclusions.** In this paper we characterize some matrix classes and investigate these matrix classes for Banach algebras. Dutta [2] used the difference operators Δ_r and $\Delta_{(r)}$ to compute some isometric spaces of the classical spaces c_0^F, c^F and ℓ_∞^F of sequences of fuzzy numbers. In a similar fashion we may use the more generalized difference operator $\Delta_r^{(s)}$. Moreover for the first time Talo and Başar [11] characterized some matrix classes involving sets of sequences of fuzzy numbers. Therefore one may find it interesting to study the results of this paper by considering the spaces involved in matrix transformations as spaces of sequences of fuzzy numbers. However there are differences between difference sequences of fuzzy numbers and complex numbers. For example, let (x_k) be a sequence of complex terms which converges to L. Then (Δx_k) converges to 0. But for the fuzzy numbers, when (X_k) converges to X (a fuzzy number), then (ΔX_k) converges to Z (a fuzzy number), where area bounded by the curve Z and the real line is double the area of the curve bounded by X and the real line. Further, the nature of the curve will be symmetric about the membership line.

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REFERENCES

[1] H. Dutta, *Characterization of certrain matrix classes involving generalized difference summability spaces*, Applied Sciences(APPS), **11** (2009), 60–67.

[2] H. Dutta, *On some isometric spaces of* c_0^F, c^F *and* ℓ_∞^F, Acta Univ. Apulensis, **19** (2009), 107–112.

[3] H. Dutta, *On some sequence spaces generated by* $\Delta_{(r)}$*- and* Δ_r*-difference of infinite matrices*, Int. J. Open Problems Compt. Math., **2** (2009), no. 4, 496–504.

[4] H. Dutta and B. S. Reddy, *On non-standard n-norm on some sequence spaces*, Int. J. Pure Appl. Math., (In Press).

[5] M. Et and T. Colak, *On generalized difference sequence spaces*, Soochow Jour. Math., **24** (1995), no. 4, 377–386.

[6] A. Esi, B.C. Tripathy and B. Sarma, *On some new type of generalized difference sequence spaces*, M. Slovaca, **57** (2007), no. 5, 475–482.

[7] H. Kizmaz, *On certain sequence spaces*, Cnad. Math. Bull., **24** (1981), no. 2, 169–176.

[8] I.J. Maddox, Elements of Functional Analysis, Camb. Univ. Press, 1970.

[9] S. Nanda, Matrix Transformations and Sequence Spaces, Miramare-Trieste, 1983.

[10] B.C. Tripathy and A. Esi, *A new type of difference sequence spaces*, Internat. J. Sci. Technol., **1** (2006), no. 1, 11–14.

[11] Ö. Talo and F. Başar, *Determination of the duals of classical sets of sequences of fuzzy numbers and related matrix transformations*, Comput. Math. Appl.,, **58** (2006), 717–733.

[12] B.C. Tripathy, A. Esi and B.K Tripathy, *On a new type of generalized difference Cesàro Sequence spaces*, Soochow J. Math., **31** (2005), no. 3, 333–340.

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