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THE HANGING CABLE PROBLEM FOR PRACTICAL APPLICATIONS

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Abstract. We investigate the 'hanging cable' problem for practical applications. We focus on determining the minimum distance between two vertical poles which will prevent a cable, hanging from the top of these poles, to touch the ground. We consider two set-ups, starting with the case of equal poles then generalizing to unequal poles. In both cases we assume that the only known quantities are the heights of the poles and the length of the cable.

1. **Introduction.** In many practical applications it is necessary to determine the relationship between the length of a cable hanging from two vertical poles, the height of the poles and the lowest distance between the cable and the ground. Such applications include electrical or telephone cables which, when hanging too low, could cause property damage or injuries and even deaths. In this paper we study this relationship, write the equations describing it and solve them exactly.

We start by looking at a specific set-up: given that a cable of length $120m$ is hanging from two poles of height $50m$, what is the minimum distance between the two poles which will prevent the cable from touching the ground? Obviously this problem can be stated in a different form involving taller poles and the lowest distance between the cable and the ground: given that a cable of length $120m$ is hanging from two poles of height $60m$, what is the minimum distance between the two poles which will prevent the cable from getting closer than $10m$ from the ground? After solving the problem in the first form, we give the general formulas for calculating the minimum distance between the poles for any cable length and poles' height. Finally we generalize this problem to the case of unequal poles and find the general formula for calculating the minimum distance between the poles for any cable length and any two vertical poles.

A perfectly flexible and inextensible cable of uniform density and cross section hanging freely from two poles assumes the shape of a catenary. The equation (up to translations and rotations) of a catenary in Cartesian coordinates has the form (see [3] section 7.7 page 464)

$$y = a\cosh\left(\frac{x}{a}\right) \tag{1}$$

where cosh is the hyperbolic cosine function. The scaling factor a is usually interpreted as the ratio between the horizontal component of the tension on the cable and the weight of the cable per unit length. For a given value of the parameter a

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the shape of the catenary is known. However, in the problem we are investigating, we are regarding a as an unknown which depends on the distance between the poles that the cable is hanging from .

In this paper we deal with the limiting case in which the catenary is tangent to the x-axis (the ground) and therefore we subtract a to arrive at the equation we will use in the problem

$$y = a\cosh\left(\frac{x}{a}\right) - a. \tag{2}$$

2. **Short history of the catenary.**

The word "catenary" comes from the latin word "catena" which means "chain". The equation describing the curve has been known and well understood for a long time.

In his last book, published in 1638, *Discourses and Mathematical Demonstrations Relating to Two New Sciences* (see [1]), Galileo made reference to the curve given by a hanging chain, and stated that a hanging chain resembles a parabola, correctly observing that this approximation improves as the curvature gets smaller and is almost exact when the elevation is less than 45°. A few years later the German mathematician Joachim Jungius proved that the shape of the hanging cable is not a parabola. However he was unable to derive the equation of the curve. His results and conclusions were published 12 years after his death, in 1669.

In 1691, three heavyweights of classical mathematical physics, Gottfried Wilhelm Leibniz, Christiaan Huygens, and Johann Bernoulli, derived the equation in response to a challenge by Jakob Bernoulli. Huygens first used the term "catenaria" in a letter to Leibniz in 1690, and David Gregory, a Scottish mathematician and astronomer, wrote a treatise on the catenary in 1690. However Thomas Jefferson is usually credited with the English word "catenary".

The hanging cable derivation arises from analyzing it in the sense of a physical problem. The only forces acting on a hanging cable at a certain point are its weight and the tension in the cable. The resultant of these forces must equal to zero considering the cable is at rest. By knowing their sum a differential equation arises with the unique solution of cosine hyperbolic. For a more detailed history and the derivation of the equation describing the shape of the catenary curve see reference [2].

3. **First case: equal poles.**

Assume that the length of the cable is $120m$ and the two poles have equal height of $50m$. We also assume that each pole is located at distance x from the midpoint which is assumed to be the y-axis (see Figure 1). With the given data, we can model the problem using two equations.

First we consider the equation which calculates half of the length of the cable using the arclength formula for the catenary described by the function in Equation (2):

$$\int_0^x \sqrt{1 + (\frac{dy}{dt})^2} dt = 60. \tag{3}$$

Second, we consider the equation which describes the height of the poles or the height of the cable at distance x from the midpoint

$$y(x) = 50. \tag{4}$$

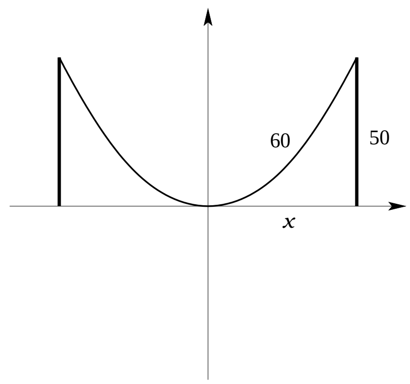

Figure 1. The hanging cable problem for equal poles: an example

After integrating Equation (3) and substituting the expression for y from Equation (2) into Equation (4) our two main equations become

$$a\sinh\left(\frac{x}{a}\right) = 60 \tag{5}$$

and

$$a\cosh\left(\frac{x}{a}\right) = 50 + a. \tag{6}$$

We arrive at this system of equations with an objective: first it is necessary to find the a value which models the hanging cable or catenary, and only then we can solve for x. To solve the system, we divide both equations by a, then square both of them and subtract them. Using the hyperbolic identity

$$\cosh^2(t) - \sinh^2(t) = 1 \tag{7}$$

we find

$$\left(\frac{50+a}{a}\right)^2 - \left(\frac{60}{a}\right)^2 = 1, \tag{8}$$

which gives

$$a = 11. \tag{9}$$

Plugging a into Equation (5) we can solve for x and find

$$x = 11\ln(11) \simeq 26.375. \tag{10}$$

The distance between the two poles when the cable is tangent to the ground is therefore

$$2x \simeq 52.75. \tag{11}$$

We can generalize this example and find a general method for calculating the distance between two equal poles given the poles' height z and the cable length $2y$ when the cable barely touches the ground (See Figure 2). As before, we can write a two equations modelling our problem:

$$a\sinh(\frac{x}{a}) = y \tag{12}$$

and

$$a\cosh(\frac{x}{a}) - a = z. \tag{13}$$

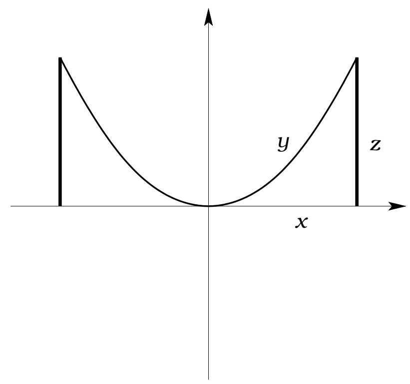

FIGURE 2. The hanging cable problem for equal poles: general case

Isolating each trigonometric hyperbolic term, squaring and subtracting, according to the identity in Equation (7), we find

$$(\frac{z+a}{a})^2 - (\frac{y}{a})^2 = 1 \tag{14}$$

Solving for a, we arrive at

$$a = \frac{y^2 - z^2}{2z}. \tag{15}$$

We can now solve for the general form of $2x$, the distance between poles, by plugging the a value from Equation (15) into Equation (12) to find

$$2x = 2a \ln \left(\frac{y}{a} + \sqrt{\left(\frac{y}{a}\right)^2 + 1} \right) \tag{16}$$

or, after using the formula for a from Equation (15),

$$2x = \frac{(y^2 - z^2)\ln(\frac{2yz}{y^2-z^2} + \sqrt{\frac{4y^2z^2}{y^4-2y^2z^2+z^4} + 1})}{z} \tag{17}$$

As a final comment we notice that if $y = 60$ and $z = 50$ formulas (15) and (17) give us $a = 11$ and $2x \simeq 52.75$. just as expected.

4. **Second case: unequal poles.** In this section we investigate a similar problem but this time with poles of different heights. Assume that the length of the cable is $140m$ and the two poles have heights of $50m$ and $70m$ (see Figure 3). Again, our goal is to determine the minimum distance between the two poles that will prevent the cable from touching the ground. With the given data, we model the problem using five equations:

$$a \sinh \frac{x_1}{a} = y_1 \tag{18}$$

and

$$a \sinh \frac{x_2}{a} = y_2 \tag{19}$$

describing the lengths of the right piece and the left piece of the cable respectively,

$$y_1 + y_2 = 140 \tag{20}$$

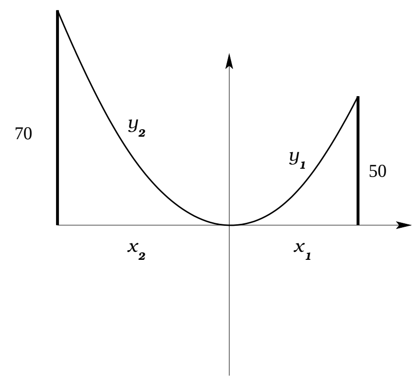

FIGURE 3. The hanging cable problem for unequal poles: an example

representing the total length of the cable, and

$$a\cosh\frac{x_1}{a} - a = 50 \tag{21}$$

and

$$a\cosh\frac{x_2}{a} - a = 70 \tag{22}$$

describing the heights of the two poles.

Using the trigonometric hyperbolic identity in Equation (7) on Equations (18) and (21) and Equations (19) and (22) we find

$$(\frac{50+a}{a})^2 - (\frac{y_1}{a})^2 = 1 \tag{23}$$

and

$$(\frac{70+a}{a})^2 - (\frac{y_2}{a})^2 = 1. \tag{24}$$

Solving for y_1 and y_2 and plugging them into Equation (20) we get

$$\sqrt{2500+100a} + \sqrt{4900+140a} = 140. \tag{25}$$

By squaring twice, we arrive at the following quadratic equation for a

$$a^2 - 5760a + 62400 = 0. \tag{26}$$

with the additional condition that $a < 50.8$. Equation (26) can be solved using the quadratic formula to find

$$a = 2880 - 280\sqrt{105} \approx 10.85 \tag{27}$$

after discarding the larger solution. With this value of a we can calculate

$$y_1 = 40\sqrt{105} - 350 \approx 59.88 \tag{28}$$

and

$$y_2 = 490 - 40\sqrt{105} \approx 80.12 \tag{29}$$

and finally

$$x_1 = 40\left(7\sqrt{105} - 72\right)\ln\left(\frac{2}{\sqrt{105}+12}\right) \approx 26.15 \tag{30}$$

and

$$x_2 = -40\left(7\sqrt{105} - 72\right)\ln\left(\frac{2\left(\sqrt{105}+12\right)}{3}\right) \approx 29.27. \tag{31}$$

We can generalize this example and find a general method for calculating the distance between two unequal poles given the poles' heights $z_1$and z_2 and the cable length y when the cable barely touches the ground (See Figure 4). As before, the

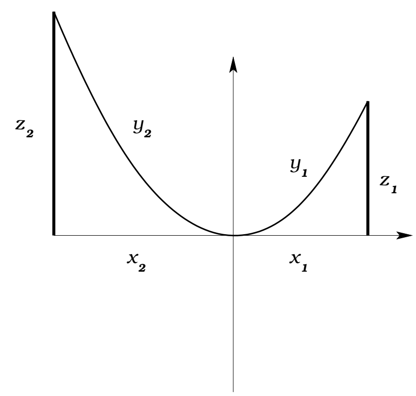

FIGURE 4. The hanging cable problem for unequal poles: general case

five equations which model this problem are:

$$a \sinh \frac{x_1}{a} = y_1 \tag{32}$$

and

$$a \sinh \frac{x_2}{a} = y_2 \tag{33}$$

describing the lengths of the right and the left pieces of the cable respectively,

$$y_1 + y_2 = y \tag{34}$$

representing the total length of the cable, and

$$a \cosh \frac{x_1}{a} - a = z_1 \tag{35}$$

and

$$a \cosh \frac{x_2}{a} - a = z_2 \tag{36}$$

describing the heights of the two poles.

Using the trigonometric hyperbolic identity in Equation (7) on Equations (32) and (35) and Equations (33) and (36) we find

$$(\frac{z_1 + a}{a})^2 - (\frac{y_1}{a})^2 = 1 \tag{37}$$

and

$$(\frac{z_2 + a}{a})^2 - (\frac{y_2}{a})^2 = 1. \tag{38}$$

Solving for y_1 and y_2 and plugging them into Equation (34) we get

$$\sqrt{z_1^2 + 2z_1a} + \sqrt{z_2^2 + 2z_2a} = y. \tag{39}$$

By squaring twice, we arrive at the following quadratic equation for a

$$4(z_1-z_2)^2a^2+4(z_1+z_2)[(z_1-z_2)^2-y^2]a+[(z_1-z_2)^2-y^2][(z_1+z_2)^2-y^2] = 0. \tag{40}$$

Notice that, in the case when the poles are of equal length, i.e. $z_1 = z_2$ and implicitly $y_1 = y_2 = \frac{y}{2}$, we can solve directly for a to obtain

$$a = \frac{\left(\frac{y}{2}\right)^2 - z^2}{2z} \tag{41}$$

as in Equation (15).

In the case when $z_1 \neq z_2$ we solve Equation (40) using the quadratic formula and find

$$a_{1,2} = \frac{(z_1+z_2)\left[y^2-(z_1-z_2)^2\right] \pm 2y\sqrt{z_1 z_2\left[y^2-(z_1-z_2)^2\right]}}{2(z_1-z_2)^2}. \tag{42}$$

Out of the two solutions found, only one (the one with a minus sign) satisfies the additional restriction on a

$$a < \frac{y^2 - z_1^2 - z_2^2}{2(z_1+z_2)} \tag{43}$$

and so the unique formula for a is

$$a = \frac{(z_1+z_2)\left[y^2-(z_1-z_2)^2\right] - 2y\sqrt{z_1 z_2\left[y^2-(z_1-z_2)^2\right]}}{2(z_1-z_2)^2}. \tag{44}$$

Notice that if $y = 140$ and $z_1 = 50$ and $z_2 = 70$ this formula gives us $a = 10.85$ which is the value found in the previous example.

With this value of a we can calculate

$$y_1 = \sqrt{z_1^2 + 2z_1 a} \tag{45}$$

and

$$y_2 = \sqrt{z_2^2 + 2z_2 a} \tag{46}$$

then (from Equations (32) and (35))

$$x_1 = a \ln \frac{z_1 + y_1 + a}{a} \tag{47}$$

and (from Equations (33) and (36))

$$x_2 = a \ln \frac{z_2 + y_2 + a}{a}. \tag{48}$$

It is easy to see that in the special case when $y = 140$, $z_1 = 50$ and $z_2 = 70$ we find, as in the previous section, $y_1 = 59.88$, $y_2 = 80.12$, $x_1 = 26.15$ and $x_2 = 29.27$.

5. **Conclusions.** In this paper we studied the 'hanging cable' problem for two cases, equal and unequal poles. We determined the minimum distance between two vertical poles, and their position with respect to the z-axis, which will prevent a cable, hanging from the top of these poles, to touch the ground. This problem has applications in engineering, e.g. the design of bridges, electrical and phone cable deployment and others.

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