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SOME CHARACTERIZATIONS OF MODULAR AND DISTRIBUTIVE JP-SEMILATTICES

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Abstract. A meet semilattice with a partial join operation satisfying some axioms is a JP-semilattice. In this paper we study the modular and distributive JP-semilattices. We give several characterizations of modular and distributive JP-semilattices. We also prove the Separation Theorem and its extension for minimal prime ideal of distributive JP-semilattices. JP-congruences, have also been studied.

1. **Introduction.** A *partial lattice* is a semilattice with a partial binary operation. Partial lattices have been studied by many authors. For the background of partial lattices we refer the reader to [3, 5, 6, 7]. In this paper we study semilattice (meet semilattice) with a partial operation. We refer the reader to [5, 6, 4] for semilattices and lattices.

An algebraic structure $\mathbf{S} = \langle S; \wedge, \vee \rangle$ where $\langle S; \wedge \rangle$ is a meet semilattice and $\vee$ is a partial binary operation on S is said to be a *join partial semilattice* (or *JP-semilattice*) if for all $x, y, z \in S$,

(i) $x \vee x = x$;
(ii) $x \vee y$ exists implies $y \vee x$ exists and $x \vee y = y \vee x$;
(iii) $x \vee y$, $y \vee z$ and $(x \vee y) \vee z$ exist imply $x \vee (y \vee z)$ exists and $(x \vee y) \vee z = x \vee (y \vee z)$;
(iv) $x \vee y$ exists implies $x = x \wedge (x \vee y)$;
(v) $y \vee z$ exists implies $(x \wedge y) \vee (x \wedge z)$ exists.

Clearly as an algebraic structure a JP-semilattice is intermediate between semilattices and lattices. A semilattice is a set with a binary operation $\wedge$ satisfying certain axioms; a lattice is a set with two binary operations $\wedge$ and $\vee$, again with certain axioms, including those of a semilattice for $\wedge$. A JP-semilattice has a semilattice operation $\wedge$, satisfying those axioms of semilattice and a partial binary operation $\vee$ satisfying (i)–(v) given above. We can represent a JP-semilattice by an

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ordered set and hence can be pictured in a Hasse diagram. Let $\mathbf{S} = \langle S; \wedge, \vee\rangle$ be a JP-semilattice. Define a binary relation $\leqslant$ on S by

$$x \leqslant y \text{ if and only if } x \wedge y = x.$$

Then $\leqslant$ is a partial ordering relation. Moreover $x \wedge y = \inf\{x, y\}$ and if $x \vee y$ exists, then $x \vee y = \sup\{x, y\}$. We can construct a picture of a relation $\leqslant$ on S in the coordinate plane. For each $x \in S$ draw a small circle. The small circle of x represent that $x \leqslant x$. If x is covered by y, then place the circle of x lower than the circle of y and take a line segment joining the circles. If $x \vee y$ does not exist but there is an upper bound u of x and y, then there are line segments from x and y to a lower point of u but there is no small circle joining the line segments. For example see Figure 1.

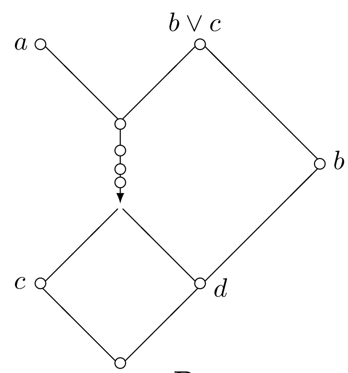

FIGURE 1. a non-JP-semilattice

Observe that not every semilattice is a JP-semilattice; for example, the semilattice $\mathbf{P}$ given in Figure 1 is not a JP-semilattice. Here $b \vee c$ exists, but $(a \wedge b) \vee (a \wedge c)$ does not.

A JP-semilattice $\mathbf{S}$ is said to be a *modular* if for all $x, y, z \in S$ such that $z \leqslant x$ and $y \vee z$ exists implies

$$x \wedge (y \vee z) = (x \wedge y) \vee z.$$

A JP-semilattice $\mathbf{S}$ is said to be a *distributive* if for all $x, y, z \in S$ such that $z \leqslant x$ and $y \vee z$ exists implies

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Clearly, the concepts of modularity and distributivity of a JP-semi-lattice $\mathbf{S}$ coincide with the concepts of modularity and distributivity when $\mathbf{S}$ is a lattice. The standard characterizations for modular and distributive lattices are given bellow.

Theorem 1. *Let* $\mathbf{L}$ *be a lattice. Then*

(a) *L is modular if and only if it has no sublattice isomorphic to the pentagonal lattice $\mathcal{N}_5$ (see Figure 2);*

(b) *L is distributive if and only if it has no sublattice isomorphic to the pentagonal lattice $\mathcal{N}_5$ or the diamond lattice $\mathcal{M}_3$ (see Figure 2);*

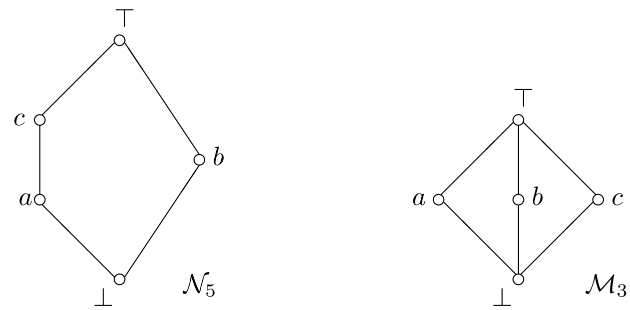

FIGURE 2. the pentagon and the diamond

We refer the reader to [4, 5] for the proof of the above theorem. Thus the pentagonal lattice $\mathcal{N}_5$ (see Figure 2) is a non-modular and hence a non-distributive JP-semilattice, and the diamond lattice $\mathcal{M}_3$ (see Figure 2) is a modular but non-distributive JP-semilattice.

In Section 2 we give a characterization of modular JP-semilattices which is a generalization of Theorem 1 (a). In Section 3 we study the ideal lattice of modular JP-semilattices. In Section 4 we study the ideal lattice of distributive JP-semilattices. Here we give a characterization of distributive JP-semilattices. Stone's Separation Theorem plays an important role in Lattice Theory. In Section 5 we generalize the Separation Theorem for distributive JP-semilattices. In Section 6 we discuss congruences of a JP-semilattice. We give a characterization of distributive JP-semilattices through a congruence.

2. **Characterizations for modular and distributive JP-semilattices.** In this section our aim is to characterize the modular and distributive JP-semilattices.

Two examples. Consider the JP-semilattices $\mathcal{N}_\infty$ and $\mathcal{M}_\infty$ given by the following diagrams (see Figure 3). The JP-semilattice $\mathcal{N}_\infty$ is said to be the *JP-pentagon* and

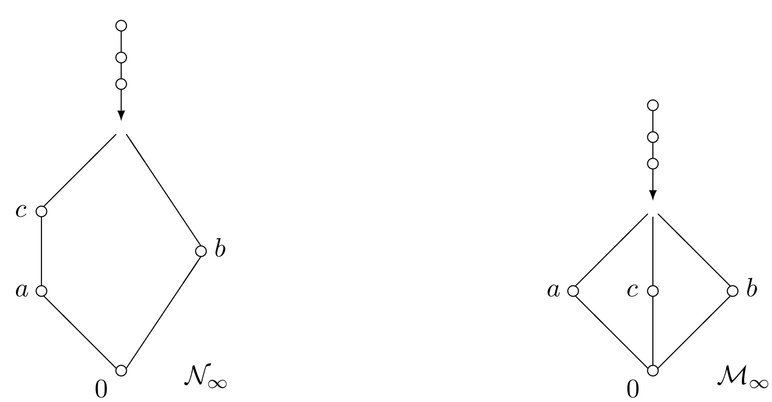

FIGURE 3. the JP-pentagon and the JP-diamond

the JP-semilattice $\mathcal{M}_\infty$ is said to be the *JP-diamond*.

Claim 2. *The JP-pentagon $\mathcal{N}_\infty$ and the JP-diamond $\mathcal{M}_\infty$ are distributive JP-semilattices.*

Proof. In both cases, if $y \vee z$ exists, then clearly, either $y \leqslant z$ or $z \leqslant y$. Without loss of generality, let $y \leqslant z$. Then $x \wedge y \leqslant x \wedge z$. Hence

$$x \wedge (y \vee z) = x \wedge z = (x \wedge y) \vee (x \wedge z).$$

□

First we have the following result.

Proposition 3. *Let* $\mathbf{S}$ *be a distributive JP-semilattice. Then for all $x, y, z \in S$ the existence of $x \vee z$ and $y \vee z$ implies the existence of $(x \wedge y) \vee z$ and*

$$(x \wedge y) \vee z = (x \vee z) \wedge (y \vee z).$$

Proof. By axiom (v) of JP-semilattices, $x \vee z$ and $y \vee z$ exist implies $((x \vee z) \wedge y) \vee ((x \vee z) \wedge z)$ exists. That is, $((x \vee z) \wedge y) \vee z$ exists. Now $x \vee z$ exists implies $(x \wedge y) \vee (z \wedge y)$ exists and $(x \wedge y) \vee (z \wedge y) = (x \vee z) \wedge y$. Hence $((x \wedge y) \vee (z \wedge y)) \vee z$ exists. That is, $(x \wedge y) \vee z$ exists. This implies

$$\begin{aligned}(x \vee z) \wedge (y \vee z) &= ((x \vee z) \wedge y) \vee ((x \vee z) \wedge z)\\ &= ((x \vee z) \wedge y) \vee z = ((x \wedge y) \vee (z \wedge y)) \vee z = (x \wedge y) \vee z.\end{aligned}$$

□

Theorem 4. *Every distributive JP-semilattice is modular but the converse is not necessarily true.*

Proof. Let $\mathbf{S}$ be a distributive JP-semilattice and let $a, b, c \in S$ such that $c \leqslant a$ and $b \vee c$ exists. Then $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) = (a \wedge b) \vee c$. Therefore $\mathbf{S}$ is modular. The diamond lattice $\mathcal{M}_3$ given in Figure 2 is a modular JP-semilattice but not distributive. □

A JP-semilattice $\mathbf{A} = \langle A; \wedge, \vee \rangle$ is said to be a *subJP-semilattice* of a JP-semilattice $\mathbf{S}$ if $A \subseteq S$ and $\wedge$ and $\vee$ in $\mathbf{A}$ are the restrictions of $\wedge$ and $\vee$ in $\mathbf{S}$.

Theorem 5. *Every subJP-semilattice of a modular (distributive) JP-semi-lattice is modular (distributive).*

Proof. Let $\mathbf{M}$ be a subJP-semilattice of a modular JP-semilattice $\mathbf{L}$. Let $a, b, c \in M$ such that $c \leqslant a$. If $b \vee c$ exists in $\mathbf{M}$, then this hold in L. Hence $(a \wedge b) \vee c$ exists in L and $a \wedge (b \vee c) = (a \wedge b) \vee c$. Since $a \wedge (b \vee c) \in M$, we have $(a \wedge b) \vee c$ exists in M and $a \wedge (b \vee c) = (a \wedge b) \vee c$. Hence M is a modular JP-semilattice.

By a similar argument we can easily show that every subJP-semilattice of a distributive JP-semilattice is distributive. □

We have the following characterization of modular JP-semilattices.

Theorem 6. *Let* $\mathbf{S}$ *be a JP-semilattice. Then* $\mathbf{S}$ *is non-modular if and only if it has a sublattice isomorphic to the pentagonal lattice*

Proof. Let $\mathbf{S}$ be non-modular. Then there exists $a, b, c \in S$ such that $c \leqslant a$ and $b \vee c$ exists, and $u = (a \wedge b) \vee c < a \wedge (b \vee c) = v$. Now $v \wedge b = (a \wedge (b \vee c)) \wedge b = a \wedge b$. Hence $u \wedge b \leqslant v \wedge b = a \wedge b \leqslant u$ and hence $a \wedge b \leqslant u \wedge b$. Therefore, $u \wedge b = a \wedge b = v \wedge b$.

Consequently, $b \vee c = (b \vee (a \wedge b)) \vee c = ((a \wedge b) \vee c) \vee b = u \vee b$. First we claim that $v \vee b$ exists. If not, then since $v, b \leqslant b \vee c$, there is an infinite chain

$b \vee c > c_1 > c_2 > \cdots$ such that $v, b \leqslant c_i$ for each i. Now $c, b \leqslant c_i$ for each i implies $b \vee c \leqslant c_i$ for each i, which is a contradiction. Hence $v \vee b$ exists. Now $v \vee b \geqslant u \vee b = b \vee c \geqslant v, b$ implies $b \vee c \geqslant v \vee b$. Thus $v \vee b = u \vee b$. Therefore $\{a \wedge b, u, v, b, b \vee c\}$ form a lattice which is isomorphic to the pentagonal lattice.

Conversely, suppose **S** is modular. Since every subJP-semilattice of a modular lattice is modular, it does not contain the pentagonal lattice as a subJP-semilattice. □

Unfortunately, we could not prove or disprove the 'standard' characterization for distributive JP-semilattices. So we have the following conjecture.

Conjecture 7. *Let* **S** *be a JP-semilattice. Then* **S** *is non-distributive if and only if it has a sublattice isomorphic to either the pentagonal lattice or the diamond lattice.*

3. **Ideals of modular JP-semilattices.** In this section we study the ideals of modular JP-semilattices. A non-empty subset I of a JP-semilattice S is said to be an *ideal* of S if

(i) $x, y \in I$ and $x \vee y$ exists, implies $x \vee y \in I$,
(ii) $x \in I$ and $y \leqslant x$ implies $y \in I$.

The set of all ideals of a JP-semilattice **S** will be denoted by $\mathcal{I}(S)$. It is easy to see that the intersection of two ideals is again an ideal, so that intersection serves as a meet operation on the set $\mathcal{I}(S)$. In order to make this set into a lattice, we need a join operation. If I and J are ideals, their join $I \vee J$ should be the least ideal to contain both the sets I and J. This motivates us to define, for any non-empty subset K of a JP-semilattice S, the smallest ideal containing K. It is denoted by $(K]$ and is called the *ideal generated by* K. For $a \in S$, the ideal $(a]$ is called the *princpal ideal* generated by a. A subset Q of a JP-semilattice S satisfying the above condition (ii) is called a down-set. The following result is trivial.

Lemma 8. *Let* **S** *be a JP-semilattice and* $\emptyset \neq K \subseteq S$. *Define* $K_0 = K$ *and for* $n \geqslant 1$,

$$K_n = \{x \in S \mid x \leqslant y \vee z \text{ for } y, z \in K_{n-1}\}.$$

Then for each $n \geqslant 1$, K_n *is a down-set and*

$$K_0 \subseteq K_1 \subseteq K_2 \subseteq \cdots .$$

The following results are consequence of the above lemma.

Theorem 9. *Let* **S** *be a JP-semilattice and* $\emptyset \neq K \subseteq S$. *Then*

(i) $(K] = \bigcup_{n=0}^{\infty} K_n$ *where* $K_0 = K$ *and for* $n \geqslant 1$,

$$K_n = \{x \in S \mid x \leqslant y \vee z \text{ for } y, z \in K_{n-1}\}$$

(ii) *For* $a \in S$ *we have* $(a] = \{x \in S \mid x \leqslant a\}$.

Proof. (i) By Lemma 8 trivially, $\bigcup_{n=0}^{\infty} K_n$ is a down-set.

Let $x, y \in \bigcup_{n=0}^{\infty} K_n$ such that $x \vee y$ exists. Then $x, y \in K_n$ for some $n \geqslant 0$. Since $x \vee y \leqslant x \vee y$ for some $x, y \in K_n$, we have $x \vee y \in K_{n+1}$. Hence $x \vee y \in \bigcup_{n=0}^{\infty} K_n$. Therefore, $\bigcup_{n=0}^{\infty} K_n$ is an ideal of S.

Let I be an ideal containing $K = K_0$. We use the mathematical induction to show that for each $n \geqslant 0$, $K_n \subset I$. Let $K_n \subseteq I$ for some $n \geqslant 1$ and let $x \in K_{n+1}$. Then $x \leqslant y \vee z$ for some $y, z \in K_n$ and hence $y \vee z \in I$ as I is an ideal. Therefore, $x \in I$. Hence for all $n \geqslant 0$, $K_n \subseteq I$. If $x \in \bigcup_{n=0}^{\infty} K_n$, then $x \in K_n$ for some $n \geqslant 0$. Hence $x \in I$. Thus $\bigcup_{n=0}^{\infty} K_n$ is the smallest ideal containing K. Hence $(K] = \bigcup_{n=0}^{\infty} K_n$.

(ii) is a particular case of (i). □

Now we clearly have the following result which gives us a description of the join of two ideals of a JP-semilattice.

Theorem 10. *Let I and J be two ideals of a JP-semilattice S. Then*

$$I \vee J = \bigcup_{n=0}^{\infty} A_n$$

where $A_0 = I \cup J$ and for $n \geqslant 1$,

$$A_n = \{x \in S \mid x \leqslant y \vee z \text{ for } y, z \in A_{n-1}\}$$

It is routine to show that $\mathcal{I}(\mathbf{S})$ is an algebraic lattice.

Remark. For any ideals I and J of a JP-semilattice $\mathbf{S}$, the description of $I \vee J$ is not as easy as for the joins in semilattices or lattices. Even $I \vee J$ can not be written simply as $\{x \leqslant y \vee z \mid y \in I, z \in J \text{ whenever } y \vee z \text{ exists}\}$. For example, consider the JP-semilattice $\mathbf{B}$ given in the Figure 4. Suppose $I = (a]$ and $J = (b]$. Then

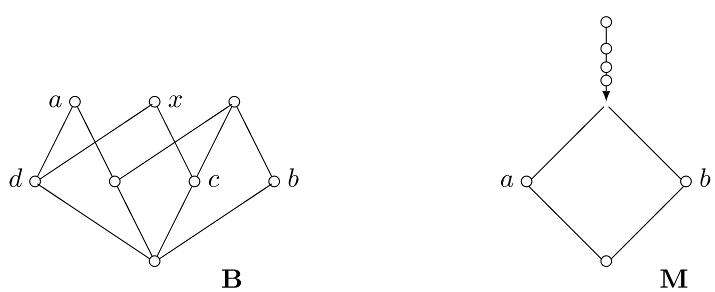

FIGURE 4. Two JP-semilattices

$x \in I \vee J$, but $x \nleqslant i \vee j$ for any $i \in I$ and $j \in J$. This observation shows that there are difficulties in studying the lattice $\mathcal{I}(S)$.

Now we turn our attention to principal ideals of a JP-semilattice. It is easy to show that the join of two principal ideals need not be principal. For example, consider the JP-semilattice $\mathbf{M}$ given in the Figure 4. Here $(a] \vee (b] = \{0, a, b\}$ is not principal. We have the following useful results.

Lemma 11. *Let* $\mathbf{S}$ *be a JP-semilattice. If* $x \vee y$ *exists, then* $(x \vee y] = (x] \vee (y]$.

Proof. We have $x, y \in (x] \cup (y]$. Hence $x \vee y \in (x] \vee (y]$. Thus $(x \vee y] \subseteq (x] \vee (y]$. The reverse inclusion is trivial. Hence $(x \vee y] = (x] \vee (y]$. □

Proposition 12. *Let* $\mathbf{S}$ *be a JP-semilattice. For any* $x, y \in S$, *we have* $(x] \vee (y]$ *is a principal ideal if and only if* $x \vee y$ *exists.*

Proof. If $x \vee y$ exists, then by the above lemma $(x] \vee (y] = (x \vee y]$ and hence $(x] \vee (y]$ is a principal ideal of S.

Conversely, let $(x] \vee (y]$ be a principal ideal. Suppose $(x] \vee (y] = (c]$. Then $x, y \leqslant c$. We show that c is the least upper bound of x and y. Suppose $x, y \leqslant d$. Then $(c] = (x] \vee (y] \subseteq (d]$. Hence $c \leqslant d$. Thus $x \vee y$ exists and $x \vee y = c$. □

Theorem 13. *Let* $\mathbf{S}$ *be a JP-semilattice. If* $\mathcal{I}(S)$ *is modular, then* $\mathbf{S}$ *is modular, but the converse is not necessarily true.*

Proof. Let $\mathcal{I}(S)$ be modular and let $x, y, z \in S$ with $z \leqslant x$. Then $(z] \subseteq (x]$. If $y \vee z$ exists, then $(x \wedge y) \vee z$ exists and

$$\begin{aligned}(x \wedge (y \vee z)] &= (x] \wedge (y \vee z] \\ &= (x] \wedge ((y] \vee (z]), \quad \text{by Lemma 11} \\ &= ((x] \wedge (y]) \vee (z], \quad \text{as } \mathcal{I}(\mathbf{S}) \text{ is modular} \\ &= (x \wedge y] \vee (z] \\ &= ((x \wedge y) \vee z]\end{aligned}$$

Thus $x \wedge (y \vee z) = (x \wedge y) \vee z$. Hence $\mathbf{S}$ is modular.

To prove the converse is not true, consider the JP-semilattice $\mathbf{B}$ in Figure 5. We call it a "butterfly". Clearly, $\mathbf{B}$ is modular as it has no sublattice isomorphic to the pentagonal lattice. Observe that the lattice $\mathcal{I}(\mathbf{B})$ contains a sublattice $\{(0], (d], (d, c], (a, b], B\}$ (see the bullet elements) which is isomorphic to the pentagonal lattice and hence $\mathcal{I}(\mathbf{S})$ is a non-modular lattice. □

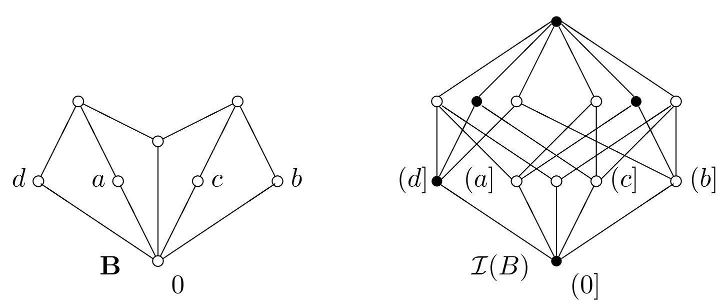

FIGURE 5. the butterfly and its lattice of ideals

Here is a characterization of modular JP-semilattices. The proof follows directly from the Lemma 11.

Theorem 14. *Let* $\mathbf{S}$ *be a JP-semilattice. Then* $\mathbf{S}$ *is modular if and only if for any* $x, y, z \in S$ *such that* $z \leqslant x$ *and* $y \vee z$ *exists implies* $(x] \wedge ((y] \vee (z]) = ((x] \wedge (y]) \vee (z]$.

4. **Ideals of distributive JP-semilattices.** We turn our attention to some characterizations of distributive JP-semilattices. First we have the following useful lemma.

Lemma 15. *Let I and J be two ideals of a distributive JP-semilattice* $\mathbf{S}$*. Then*

$$I \vee J = \bigcup_{n=0}^{\infty} A_n$$

where $A_0 = I \cup J$ and for $n \geqslant 1$, and

$$A_n = \{x \in S \mid x = y \vee z \text{ for } y, z \in A_{n-1}\}.$$

Proof. Since $\mathbf{S}$ is distributive, this is a consequence of Theorem 10. □

The following results give characterizations of distributive JP-semilattices.

Theorem 16. *Let I and J be two ideals of a JP-semilattice* $\mathbf{S}$*. Then the following are equivalent:*

(a) $\mathbf{S}$ *is distributive;*
(b) $I \vee J = \{a_1 \vee a_2 \vee \cdots \vee a_n \mid a_i \in I \cup J \text{ for all } i = 1, 2, \cdots, n\}$*;*
(c) $\mathcal{I}(S)$ *is a distributive lattice;*
(d) *for any $x, y, z \in S$ for which $y \vee z$ exists,*

$$(x] \wedge ((y] \vee (z]) = ((x] \wedge (y]) \vee ((x] \wedge (z]).$$

Proof. (a) ⇒ (b). By using mathematical induction to extend Lemma 15.

(b) ⇒ (c). Let $I, J, K \in \mathcal{I}(S)$ and $x \in I \cap (J \vee K)$. Then $x \in I$ and $x = a_1 \vee a_2 \vee \cdots \vee a_n$ where $a_i \in J \cup K$ for all $i = 1, 2, \cdots, n$. Now for each $i = 1, 2, \cdots, n$, we have $a_i \leqslant x$ and hence $a_i \in I \cap J$ or $I \cap K$. Hence $a_i \in (I \cap J) \cup (I \cap K)$. Therefore, $x \in (I \cap J) \vee (I \cap K)$. The reverse inclusion is trivial and hence $\mathcal{I}(S)$ is a distributive lattice.

(c) ⇒ (d). Trivial.

(d) ⇒ (a). Let $x, y, z \in S$ with $y \vee z$ exists. Then

$$\begin{aligned}(x \wedge (y \vee z)] &= (x] \cap ((y] \vee (z]) \\ &= ((x \cap (y]) \vee ((x] \cap (z]) \\ &= (x \wedge y] \vee (x \wedge z] \\ &= ((x \wedge y) \vee (x \wedge z)].\end{aligned}$$

Hence $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$. Therefore, $\mathbf{S}$ is distributive. □

Let $\mathbf{S}$ be a JP-semilattice. The set of all down-subsets of S is denoted by $\mathcal{O}(S)$. It is evident that $\mathcal{O}(S)$ is a bounded complete distributive lattice for any ordered set $\mathbf{S}$ when partially ordered by set inclusion. The meet and join in $\mathcal{O}(S)$ are given by set-theoretic intersection and union respectively.

The following result clearly holds from Theorem 16.

Lemma 17. *Let* $\mathbf{S}$ *be a distributive JP-semilattice and $K \in \mathcal{O}(\mathbf{S})$. Then*

$$(K] = \{x_1 \vee x_2 \vee \cdots \vee x_n \mid x_i \in K \text{ for each } i = 1, 2, \cdots, n\}.$$

The following theorem is a generalization of [2, Theorem 2.3].

Theorem 18. *Let* $\mathbf{S}$ *be a JP-semilattice. For any $A, B, C \in \mathcal{O}(\mathbf{S})$ the following conditions are equivalent:*

(a) $\mathbf{S}$ *is distributive;*

(b) $(A] = \{a_1 \vee a_2 \vee \cdots \vee a_n \mid a_1, a_2, \cdots, a_n \in A\}$;
(c) $A \cap (B] \subseteq (A \cap B]$;
(d) $(A \cap B] = (A] \cap (B]$;
(e) $(A \cap (B \cap C]] = ((A \cap B] \cap C]$;
(f) *The map* $\varphi : \mathcal{O}(S) \to I(S)$ *defined by* $\varphi(A) = (A]$ *is an onto lattice-homomorphism.*

Proof. (a) $\Rightarrow$ (b). By Lemma 17.

(b) $\Rightarrow$ (c). Let $x \in A \cap (B]$. Then $x \in A$ and by (b), $x = b_1 \vee b_2 \vee \cdots \vee b_n$ where $b_1, b_2, \cdots, b_n \in B$. Since $A \in \mathcal{O}(S)$ and $b_i \leqslant x$ for all $i = 1, 2, \cdots, n$ we have $b_i \in A$ for all $i = 1, 2, \cdots, n$. Hence $b_i \in A \cap B$ for all $i = 1, 2, \cdots, n$. Therefore, $x \in (A \cap B]$.

(c) $\Rightarrow$ (d). By (c), we have $(A] \cap (B] \subseteq ((A] \cap B] \subseteq (A \cap B]$ for any $A, B \in \mathcal{O}(S)$. Since $(A \cap B] \subseteq (A] \cap (B]$, we have $(A \cap B] = (A] \cap (B]$. Thus (d) holds.

(d) $\Rightarrow$ (e). Suppose (d) holds. Then

$$\begin{aligned}(A \cap (B \cap C]] &= (A] \cap (B \cap C] = (A] \cap ((B] \cap (C])\\ &= ((A] \cap (B]) \cap (C] = (A \cap B] \cap (C] = ((A \cap B] \cap C].\end{aligned}$$

Thus (e) holds.

(e) $\Rightarrow$ (d). This is trivial.

(d) $\Rightarrow$ (f). For any $A, B \in \mathcal{O}(S)$, we have

$$\begin{aligned}(A] \vee (B] &= \{x_1 \vee x_2 \vee \cdots \vee x_n \mid x_1, x_2, \cdots, x_n \in (A] \cup (B]\}\\ &= \{x_1 \vee x_2 \vee \cdots \vee x_n \mid x_1, x_2, \cdots, x_n \in A \cup B\}\\ &= (A \cup B].\end{aligned}$$

Hence by (d), φ is a lattice homomorphism. Let $I \in \mathcal{I}(S)$. Then $\varphi(I) = (I] = I$. Thus φ is an onto lattice homomorphism. Therefore, (f) holds.

(f) $\Rightarrow$ (a). Since $\mathcal{O}(S)$ is always a distributive lattice, $I(S)$ is a distributive and hence by Theorem 16, S is distributive. □

5. **The Separation Theorem.** Let $\mathbf{S}$ be a JP-semilattice. A non-empty subset F of S is said to be a *filter* (or *dual ideal*) if

(i) for $x \in F$ and $y \in S$ with $x \leqslant y$ implies $y \in F$, and
(ii) for $a, b \in F$ implies $a \wedge b \in F$.

An ideal P of a JP-semilattice $\mathbf{S}$ is said to be *prime* if $a, b \in S$ such that $a \wedge b \in P$ implies either $a \in P$ or $b \in P$. A prime ideal P is called *minimal* if whenever there is a prime ideal Q such that $Q \subseteq P$, then $P = Q$.

Lemma 19. *Let* $\mathbf{S}$ *be a JP semilattice. An ideal (filter)* P *is prime if and only if* $S \setminus P$ *is a prime filter (ideal).*

Proof. Let P be a prime ideal. If $x, y \in S \setminus P$, then $x, y \notin P$. Hence $x \wedge y \notin P$ which implies $x \wedge y \in S \setminus P$. Let $x \in S \setminus P$ and $x \leqslant y$. Then $x \notin P$ and hence $y \notin P$. Therefore $y \in S \setminus P$. This implies $S \setminus P$ is a filter. Let $x, y \in S$ such that $x \vee y$ exists and $x \vee y \in S \setminus P$. Then $x \vee y \notin P$. This implies either $x \notin P$ or $y \notin P$ and consequently, either $x \in S \setminus P$ or $y \in S \setminus P$. Hence $S \setminus P$ is a prime filter.

By a reverse argument we have the converse of the statement. □

Now we have the following Separation Theorem for distributive JP-semilattice.

Theorem 20 (The JP-Separation Theorem)**.** *Let* **S** *be a JP-semilattice. Then the following are equivalent:*

(a) **S** *is distributive;*
(b) *For any ideal I and any filter F of* **S** *such that $I \cap F = \emptyset$, there exists a prime ideal P containing I such that $P \cap F = \emptyset$.*

Proof. (a) $\Rightarrow$ (b). Let $\mathcal{I}$ be the set of all ideals containing I, but disjoint from F. Then $\mathcal{I} \neq \emptyset$ as $I \in \mathcal{I}$. Let $\mathcal{C}$ be a chain in $\mathcal{I}$ and let $M := \cup\{X \mid X \in \mathcal{C}\}$. We claim that M is the maximum element in $\mathcal{C}$.

Let $x \in M$ and $y \leqslant x$. Then $x \in X$ for some $X \in \mathcal{C}$. Hence $y \in X$ as X is an ideal. Therefore $y \in M$. Let $x, y \in M$. Then $x \in X$ and $y \in Y$ for some $X, Y \in \mathcal{C}$. Since $\mathcal{C}$ is a chain, either $X \subseteq Y$ or $Y \subseteq X$. Suppose $X \subseteq Y$. So $x, y \in Y$. If $x \vee y$ exists, then $x \vee y \in Y$ as Y is an ideal. Hence $x \vee y \in M$. Moreover, M contains I and $F \cap M = \emptyset$. Therefore, M is the maximum element in $\mathcal{C}$.

Thus by Zorn's Lemma, $\mathcal{I}$ has a maximal element, say P. We claim that P is prime. If P is not prime, there exists $a, b \in S$ such that $a, b \notin P$ but $a \wedge b \in P$. Then $(P \vee (a]) \cap F \neq \emptyset$ and $(P \vee (b]) \cap F \neq \emptyset$ as P is maximal. Hence there exists $x, y \in F$ such that $x \wedge y \in (P \vee (a]) \cap (P \vee (b]) = P \vee ((a] \wedge (b])$ as **S** is a distributive lattice implies $\mathcal{I}(S)$ is a distributive lattice. Thus $x \wedge y \in F$ and $x \wedge y \in P \vee (a \wedge b] = P$ which is a contradiction to $P \cap F = \emptyset$. Hence P is a prime ideal.

(b) $\Rightarrow$ (a). Let $a, b, c \in S$ such that $b \vee c$ exists. If $(a \wedge b) \vee (a \wedge c) \neq a \wedge (b \vee c)$, then $(a \wedge b) \vee (a \wedge c) < a \wedge (b \vee c)$. Consider $I = ((a \wedge b) \vee (a \wedge c)]$ and $F = [a \wedge (b \vee c))$. Then $I \cap F = \emptyset$ and hence by (b), there is a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$. Thus $(a \wedge b) \vee (a \wedge c) \in P$, this implies $a \wedge b \in P$ and $a \wedge c \in P$. So, either $a \in P$ or $b \vee c \in P$. Hence $a \wedge (b \vee c) \in P$ which is a contradiction. Therefore, $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$. Hence S is distributive. □

Corollary 21. *Let* **S** *be a distributive JP-semilattice and let I be an ideal of S. If $a \notin I$, then there exists a prime ideal P containing I such that $a \notin P$.*

The following useful result is a generalization of a well known result of lattice theory.

Theorem 22. *Let* **S** *be a distributive JP-semilattice. Then every ideal of S is the intersection of all prime ideals containing it.*

Proof. Let **S** be a JP-semilattice and let J be an ideal of **S**. We shall show that

$$J = \bigcap\{P \mid P \text{ is a prime ideal of } S \text{ and } J \subseteq P\}.$$

Clearly, $J \subseteq$ R.H.S. If $J \neq$ R.H.S., then there is $x \in$R.H.S. such that $x \notin J$. Hence by the Separation Theorem, there is a prime ideal Q of S such that $J \subseteq Q$ and $x \notin Q$, which is a contradiction. □

The following theorem is a characterization of a minimal prime ideal containing an ideal. This is also a generalization of [8, Lemma 3.1]

Theorem 23. *Let* **S** *be a distributive JP-semilattice and let J be an ideal of S. Then a prime ideal P containing J is a minimal prime ideal containing J if and only if for each $x \in P$ there is $y \in S \setminus P$ such that $x \wedge y \in J$.*

Proof. Let P be a prime ideal of S containing J such that the given condition holds. We shall show that P is a minimal prime ideal containing J. Let K be a prime ideal containing J such that $K \subseteq P$. Let $x \in P$. Then there is $y \in S \setminus P$ such that $x \wedge y \in J$. Hence $x \wedge y \in K$ as K contains J. Since K is prime and $y \notin K$ implies $x \in K$. Hence $P \subseteq K$. Thus $K = P$. Therefore P is a minimal prime ideal containing J.

Conversely, let P be a minimal prime ideal containing J. Let $x \in P$. Suppose for all $y \in S \setminus P$, $x \wedge y \notin J$. Set $D = (S \setminus P) \vee [x)$. We claim that $0 \notin D$. For if $0 \in D$, then $0 = q \wedge x$ for some $q \in S \setminus P$. Thus, $x \wedge q = 0 \in J$ which is a contradiction. Therefore, $0 \notin D$. Then by the JP-separation Theorem 20, there is a prime filter Q such that $D \subseteq Q$ and $0 \notin Q$. Let $M = S \setminus Q$. Then by Lemma 19, M is a prime ideal. We claim that $M \cap D = \emptyset$. If $a \in M \cap D$, then $a \in M$ and hence $a \notin Q$. Thus $a \notin D$ which is a contradiction. Hence $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus P) = \emptyset$ and hence $M \subseteq P$. Also $M \neq P$, because $x \in D$ implies $x \in Q$ and hence $x \notin M$ but $x \in P$. This shows that P is not minimal which is a contradiction. Hence the given condition holds. □

Theorem 24. *Let* **S** *be a JP-semilattice with* 0 *and let P be a prime ideal of S. Let $\mathcal{C}$ be a chain of all prime ideals X of S such that $X \subseteq P$. Then*

$$Q = \bigcap \{X \subseteq P \mid X \in \mathcal{C}\}$$

is a prime ideal and hence it is a minimal prime ideal.

Proof. Clearly, $\mathcal{C}$ is non-empty as $P \in \mathcal{C}$ and Q is non-empty as $0 \in Q$. Obviously, Q is an ideal. To show that Q is prime, let $x \wedge y \in Q$. Suppose $x \notin Q$. This implies $x \notin X$ for some $X \in \mathcal{C}$. Now $x \wedge y \in Q$ implies $x \wedge y \in X$. Hence $y \in X$ as X is prime. We claim that $y \in Q$. If not, then $y \notin Y$ for some $Y \in C$ with $Y \subset X$. But $x \wedge y \in Q$ implies $x \wedge y \in Y$. Thus $x \in Y$ and so $x \in X$ as $Y \subset X$ which gives a contradiction. Therefore $y \in Q$. Hence Q is prime and in fact it is a minimal prime ideal. □

Thus we have the following extension of Stone's Separation Theorem.

Theorem 25. *Let J be an ideal and D be a filter of a distributive JP-semilattice* **S** *such that $J \cap D = \emptyset$. Then there exists a minimal prime ideal Q containing J such that $Q \cap D = \emptyset$.*

Proof. Let J be an ideal and D be a filter of a distributive JP-semilattice **S** such that $J \cap D = \emptyset$. Then by the JP-version of Stone's Separation Theorem 20, there exists a prime ideal P containing J such that $P \cap D = \emptyset$. Choose any chain $\mathcal{C}$ of prime ideals X containing J such that $X \subseteq P$. Let $Q = \cap\{X \in \mathcal{C}\}$. Then by Theorem 24, Q is a minimal prime ideal containing J and $Q \cap D = \emptyset$. □

Let **S** be a JP-semilattice with 0 and let Q be a prime ideal of S. Define

$$O(Q) := \{x \in S \mid x \wedge y = 0 \text{ for some } y \in S \setminus Q\}.$$

The following theorem is a generalization of [1, Proposition 2.2]

Theorem 26. *Let* **S** *be a distributive JP-semilattice with* 0 *and let Q be a prime ideal of S. Then*

$$O(Q) = \bigcap \{P \mid P \textit{ is a minimal prime ideal of } S \textit{ such that } P \subseteq Q\}.$$

Proof. Suppose

$$X = \bigcap \{P \mid P \text{ is a minimal prime ideal of } S \text{ such that } P \subseteq Q\}.$$

Let $x \in O(Q)$. Then $x \wedge y = 0$ for some $y \notin Q$. Let P be a minimal prime ideal contained in Q. Clearly, $y \notin P$. Since $x \wedge y = 0 \in P$ and P is prime, we have $x \in P$. Hence $x \in X$.

Conversely let $x \in X$. If $x \notin O(Q)$. Then $x \wedge y \neq 0$ for any $y \in S \setminus Q$. Let $D = [x) \vee (S \setminus Q)$. Then $0 \notin D$. For if $0 \in D$, then $x \wedge q = 0$ for some $q \in S \setminus Q$ which is a contradiction. Therefore, $0 \notin D$. Consequently, there is a minimal prime ideal M such that $M \cap D = \emptyset$. Therefore, $M \cap (S \setminus Q) = \emptyset$. Hence $M \subseteq Q$. Also $M \neq Q$ because $x \in Q$. But $x \in D$ implies $x \notin M$. This shows that there is a minimal prime ideal $M \subset Q$ such that $x \notin M$ which is a contradiction to fact that $x \in X$. Hence $x \in O(Q)$. □

6. **JP-Congruences.** Let **S** be a JP-semilattice. A semilattice congruence θ on S is said to be a *JP-congruence* if $a \equiv b(\theta)$ and $c \equiv d(\theta)$ implies that $a \vee c \equiv b \vee d(\theta)$ whenever $a \vee c$ and $b \vee d$ exist. If a relation θ satisfies the condition, then we say that θ is *compatible* with existing $\vee$

Let **S** be a distributive JP-semilattice and I be an ideal of **S**. Then by [3], the relation $\Theta(I)$ on S defined by

$$x \equiv y(\Theta(I)) \Leftrightarrow (x] \vee I = (y] \vee I$$

is a JP-congruence having I as a class. Moreover if each JP-congruence is compatible with any finite existing $\vee$, then $\Theta(I)$ is the the smallest JP-congruence having I as a class. We will use Θ_x for $\Theta((x])$.

As in Cornish and Hickman [3], the following technicality will be used frequently without explicit reference.

Let **S** *be a JP-semilattice, θ be a JP-congruence and E be an equivalence relation on S. Then to show that $\theta \subseteq E$ it is enough to show that $(a, b) \in E$ for every pair (a, b) such that $a \leqslant b$ and $a \equiv b(\theta)$*

Now we have the following result.

Theorem 27. *Let* **S** *be a distributive JP-semilattice and let I be an ideal of S. Then $R(I)$ defined by*

$$x \equiv y(R(I)) \Leftrightarrow x \wedge a \in I \text{ equivalent to } y \wedge a \in I \text{ for every } a \in S,$$

is the largest JP-congruence having I as a class.

Proof. Clearly, $R(I)$ is a semilattice congruence having I as a class. Suppose $x \equiv y(R(I))$ and $s \equiv t(R(I))$. If $x \vee s$ and $y \vee t$ exist, then for any $a \in S$ we have

$$\begin{aligned}(x \vee s) \wedge a \in I &\Leftrightarrow (x \wedge a) \vee (s \wedge a) \in I, \text{ as } \mathbf{S} \text{ is distributive} \\ &\Leftrightarrow x \wedge a, s \wedge a \in I \\ &\Leftrightarrow y \wedge a, t \wedge a \in I \\ &\Leftrightarrow (y \wedge a) \vee (t \wedge a) \in I \\ &\Leftrightarrow (y \vee t) \wedge a \in I, \text{ as } \mathbf{S} \text{ is distributive.}\end{aligned}$$

Hence $x \vee s \equiv y \vee t(R(I))$. Thus $R(I)$ is a JP-congruence. Let Γ be a JP-congruence having I as a class and $x \equiv y(\Gamma)$. Then for any $a \in S$, we have $x \wedge a \in I \Leftrightarrow y \wedge a \in I$, since $x \wedge a \equiv y \wedge a(\Gamma)$. Hence $x \equiv y(R(I))$. □

The congruence relation $\Theta(a, b)$ is the smallest congruence containing $\{a, b\}$ as a class. We have a description of $\Theta(a, b)$.

Theorem 28. *Let* $\mathbf{S}$ *be a distributive JP-semilattice and* $a, b, x, y \in S$ *such that* $a \leqslant b$. *Then*

$$x \equiv y(\Theta(a,b)) \Leftrightarrow x \wedge a = y \wedge a \text{ and } (x] \vee (b] = (y] \vee (b].$$

Proof. Let ψ denote the binary relation on S such that

$$x \equiv y(\psi) \Leftrightarrow x \wedge a = y \wedge a \text{ and } (x] \vee (b] = (y] \vee (b].$$

Then clearly ψ is an equivalence relation. Now let $x \equiv y(\psi)$ and $s \equiv t(\psi)$. Then $x \wedge a = y \wedge a$, $(x] \vee (b] = (y] \vee (b]$, $s \wedge a = t \wedge a$ and $(s] \vee (b] = (t] \vee (b]$. Hence $(x \wedge s) \wedge a = (y \wedge t) \wedge a$ and since $\mathbf{S}$ is distributive implies $\mathcal{I}(S)$ is distributive, so

$$\begin{aligned}(x \wedge s] \vee (b] &= ((x] \wedge (s]) \vee (b] = ((x] \vee (b]) \wedge ((s] \vee (b]) \\ &= ((y] \vee (b]) \wedge ((t] \vee (b]) = ((y] \wedge (t]) \vee (b] = (y \wedge t] \vee (b].\end{aligned}$$

Thus $x \wedge s \equiv y \wedge t(\psi)$. Also if $x \vee s$ and $y \vee t$ exists, then since $\mathbf{S}$ is distributive,

$$(x \vee s) \wedge a = (x \wedge a) \vee (s \wedge a) = (y \wedge a) \vee (t \wedge a) = (y \vee t) \wedge a$$

and

$$\begin{aligned}(x \vee s] \vee (b] &= ((x] \vee (s]) \vee (b] = ((x] \vee (b]) \vee ((s] \vee (b]) \\ &= ((y] \vee (b]) \vee ((t] \vee (b]) = ((y] \vee (t]) \vee (b] = (y \vee t] \vee (b].\end{aligned}$$

Thus $x \vee s \equiv y \vee t(\psi)$. Therefore, ψ is a JP-congruence. Clearly $a \equiv b(\psi)$. Let Γ be a congruence on S such that $a \equiv b(\Gamma)$. Let $x \equiv y(\psi)$ with $x \leqslant y$. Then $x \wedge a = y \wedge a$ and $(x] \vee (b] = (y] \vee (b]$. Since $a \equiv b(\Gamma)$ so, $x \wedge a \equiv x \wedge b(\Gamma)$ and $y \wedge a \equiv y \wedge b(\Gamma)$. Thus $x \wedge b \equiv x \wedge a(\Gamma) = y \wedge a \equiv y \wedge b(\Gamma)$. Now we have

$$(y] = (y] \wedge ((y] \vee (b]) = (y] \wedge ((x] \vee (b]) = ((y] \wedge (x]) \vee ((y] \wedge (b]) = (x] \vee (y \wedge b].$$

This shows that $(x] \vee (y \wedge b]$ is a principal ideal and hence by Theorem 12 we have $y = x \vee (y \wedge b) \equiv x \vee (x \wedge b)(\Gamma) = x$. Hence ψ is the smallest congruence.

Therefore, $\psi = \Theta(a, b)$. □

Let $\mathbf{S}$ and $\mathbf{P}$ be two JP-semilattices. A semilattice homomorphism $\varphi : \mathbf{S} \to \mathbf{P}$ is said to be a *JP-homomorphism* if for all $x, y \in S$ such that $x \vee y$ exists in S implies $\varphi(x) \vee \varphi(y)$ exists in P and

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y).$$

Let $\varphi : \mathbf{S} \to \mathbf{P}$ be a JP-homomorphism. The *kernel* of φ is denoted by $\ker \varphi$ and defined by

$$\ker \varphi = \{(x, y) \in S^2 \mid \varphi(x) = \varphi(y)\}.$$

Lemma 29. *Let* $\varphi : \mathbf{S} \to \mathbf{P}$ *be a JP-homomorphism. Then* $\ker \varphi$ *is a JP-congruence on* S.

Proof. Clearly $\ker \varphi$ is an equivalence relation on S. Let $x_1 \equiv y_1(\ker \varphi)$ and $x_2 \equiv y_2(\ker \varphi)$. Then $\varphi(x_1) = \varphi(y_1)$ and $\varphi(x_2) = \varphi(y_2)$. Now $\varphi(x_1 \wedge x_2) = \varphi(x_1) \wedge \varphi(x_2) = \varphi(y_1) \wedge \varphi(y_2) = \varphi(y_1 \wedge y_2)$. Therefore, $x_1 \wedge x_2 \equiv y_1 \wedge y_2(\ker \varphi)$. To prove $\ker \varphi$ is conditional compatible with $\vee$, let $x_1 \vee x_2$ and $y_1 \vee y_2$ exist. Then by the definition of a JP-homomorphism, $\varphi(x_1) \vee \varphi(x_2)$ and $\varphi(y_1) \vee \varphi(y_2)$ exist and $\varphi(x_1 \vee x_2) = \varphi(x_1) \vee \varphi(x_2)$ and $\varphi(y_1 \vee y_2) = \varphi(y_1) \vee \varphi(y_2)$. Hence $\varphi(x_1 \vee x_2) = \varphi(x_1) \vee \varphi(x_2) = \varphi(y_1) \vee \varphi(y_2) = \varphi(y_1 \vee y_2)$. Thus $x_1 \vee x_2 \equiv y_1 \vee y_2(\ker \varphi)$.

Therefore $\ker \varphi$ is a JP-congruence. □

We have the following important result for distributive JP-semilattices.

Theorem 30. *Let* **S** *be a JP-semilattice. The following conditions are equivalent:*

(a) **S** *is distributive;*

(b) *for* $a \in S$*, the map* $\varphi : S \mapsto (a]$ *given by*

$$\varphi(x) = a \wedge x$$

is a JP-homomorphism of **S** *onto* $(a]$*;*

(c) *for* $a \in S$*, the binary relation* Θ_a *on* S *defined by*

$$x \equiv y(\Theta_a) \Longleftrightarrow x \wedge a = y \wedge a$$

is a congruence relation.

Proof. (a) $\Rightarrow$ (b). Let **S** be a distributive JP-semilattice. Then for any $x, y \in S$ we have

$$\varphi(x \wedge y) = a \wedge (x \wedge y) = (a \wedge x) \wedge (a \wedge y) = \varphi(x) \wedge \varphi(y).$$

Also if $x \vee y$ exists, then

$$\varphi(x \vee y) = a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y) = \varphi(x) \vee \varphi(y).$$

Thus φ is a JP-homomorphism. If $x \in (a]$, then $x \leqslant a$ and hence $x = a \wedge x = \varphi(x)$. Therefore, (b) holds.

(b) $\Rightarrow$ (c). Define a relation Θ_a on S given by $x \equiv y(\Theta_a) \Longleftrightarrow a \wedge x = a \wedge y$. If $\varphi : x \mapsto a \wedge x$ is a map from **S** to $(a]$, then we have $x \equiv y(\Theta_a) \Longleftrightarrow \varphi(x) = \varphi(y)$. Thus $\Theta_a = \ker \varphi$. Since by (b), φ is a JP-homomorphism, so by Lemma 29, $\ker \varphi$ is a congruence. Hence Θ_a is a congruence. Thus (c) holds.

(c) $\Rightarrow$ (a). Let $x, y \in S$ with $x \vee y$ exists. Then for any $a \in S$, we have $(a \wedge x) \vee (a \wedge y)$ exists. Since $a \wedge x = a \wedge (a \wedge x)$, so $x \equiv a \wedge x(\Theta_a)$. Similarly, $y \equiv a \wedge y(\Theta_a)$. Thus $x \vee y \equiv (a \wedge x) \vee (a \wedge y)(\Theta_a)$. Hence

$$a \wedge (x \vee y) = a \wedge (a \wedge x) \vee (a \wedge y) = (a \wedge x) \vee (a \wedge y).$$

Thus (a) holds. □

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