Atlantic Electronic
Journal of Mathematics
Volume **3**, Number **1**, Winter **2008**

http://aejm.ca
http://aejm.ca/rema
pp. **30–45**

DIRECTIONAL PHUTBALL

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Abstract. There are few results pertaining to John Conway's game of phutball or *Philosophers Football*. The problem of deciding whether a given position has a winning jump has been shown to be NP-complete and for the 1-dimensional case, only a partial strategy has been found. I consider a restricted version of phutball called directional phutball, in which, as the name suggests, each player may only jump in a specified direction. The goal of this paper is to determine a partial order for the $2 \times n$ case.

1. **Introduction.** phutball, short for *Philosophers football*, is a game invented by John Conway [2] Vol 2. It is played on a Go board with a set of white stones (the men) and a single black stone (the ball) initially placed in the centre of the board. We call the right hand side of the board *Right's goal line*, and the left hand side *Left's goal line*. On his or her turn each player may either:

- Place a man (which are held common to both players) in any intersection on the board
- Jump a series of strings of placed men with the ball in any direction (the segments of a jump may be vertical, horizontal or diagonal) removing the men he or she has jumped while doing so.

The ultimate aim of phutball is to jump onto, or over the opponent's goal line (that is, either onto, or over the leftmost or rightmost goaline) which concludes the game. Note that a player may jump onto (but not over) and out of his own goal line during a move.

Figure 1 is a excerpt from [2] showing the standard first 5 moves of a game on a 15×19 board with Left placing the first man ①. Note the leftmost and rightmost vertical lines are Left and Rights goalines respectively.

Because repeated positions may occur, phutball is a loopy game. That is, in theory, play could continue indefinitely. In practice however, play ends after a finite number of moves.

Despite being defined more than 30 years ago, there are very few results pertaining to phutball; Demaine, Demaine, & Eppstein have shown that the question of whether the next player in a game of phutball has a winning move is NP-complete [3]. Even the simplified game of 1-dimensional phutball (a game of phutball played on a single horizontal strip) proves to be difficult: most reasonable heuristics are easily shown to lead to losing moves. For instance, the intuitive assumption that your position can only be improved by placing a man between the ball and your goal line is easily shown to be false [4]. There even are positions in

2000 *Mathematics Subject Classification.* Primary: 91A46 .
Key words and phrases. Combinatorial Game Theory, Philosopher's Football, atomic weight.

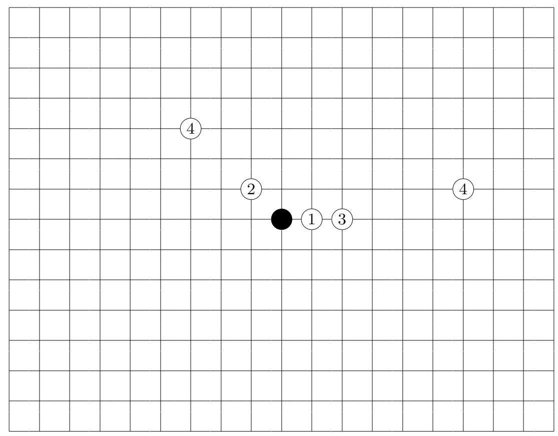
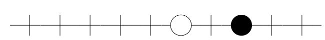


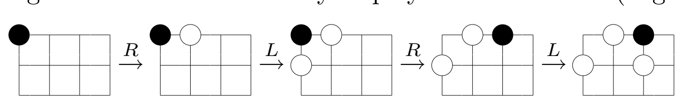
FIGURE 1. The opening moves for Phutball. Source [2].

which the only winning move is to jump backwards [4]. However, most positions where a counter intuitive move is required involve a stone placed at an even distance from the ball. This off-parity consideration also occurs in the full game and even has its own term, *poultry*. An example of a poultry move in one dimensional PHUTBALL is shown below:



Siegel [5] cites the work of Nowakowski, Ottaway and Siegel concerning the intricate loopy structure of one-dimensional PHUTBALL for boards of length ≤ 11. However, if the players are constrained to play 'efficiently', i.e. no poultry moves such as the one in the example above (called ODDISH PHUTBALL), then Grossman & Nowakowski [4] provide a complete strategy for the 1-dimensional game.

DIRECTIONAL PHUTBALL is a loose derivation of PHUTBALL aimed at eliminating the loopiness of the latter thus making analysis easier. The game is played on an $m \times n$ Go board. Initially, a black stone (the ball) is placed in the upper left hand corner of the board. On his or her turn, each player can either place a white stone (a man) on any intersection on the board, or jump a series of strings of men with the ball. However, Left may only jump vertically and Right may only jump horizontally. As such, Left wins either by jumping into the bottomost horizontal line (the *mth* row), or off of the board to the bottom. Right wins either by jumping into the the rightmost vertical line (the *nth* column), or off of the board to the right. For simplicity, we include the additional rule that no men may be placed above or to the left of the ball. A game on a 3×4 board may be played out as follows (Right plays first):



We will use the notation $P(m,n)$ to denote a Directional PHUTBALL game of size $m \times n$, m being the number of horizontal lines (rows) on the board and n being the number of vertical lines (columns). $P'(m,n)$ is the notation that will be used to denote any game $P(m,n)$ in progress and $P_i(m,n)$ a particular game in progress. The stone Ⓡ will be used to denote a placement of a man by Right, and Ⓛ to denote

Left's placements.

A few salient facts about combinatorial game theory are presented here. For more details see [1] and [2]. First, a combinatorial game is an alternating turn taking game with no elements of chance. There are two players, called Left and Right. In this paper, Left is regarded as female and Right as male. The normal play convention, which is the convention in this paper, is that the last player to move is the winner—equivalently, the first player who cannot move is the loser. There are four possibilities, or *outcome classes* under which a game, or game position can fall:

$\mathcal{N}$ **or Fuzzy:** The player whose turn it is $\mathcal{N}$ext (or the 1st to play) can win.
$\mathcal{P}$ **or Zero:** The $\mathcal{P}$revious player to have taken his turn (or the 2nd to play) can win.
$\mathcal{L}$ **or Left:** $\mathcal{L}$eft can win playing first or second.
$\mathcal{R}$ **or Right:** $\mathcal{R}$ight can win playing first or second

We refer to Left-win games (games in $\mathcal{L}$) as being *positive* and Right-win games (games in $\mathcal{R}$) as being *negative*. We call $\mathcal{P}$ and $\mathcal{N}$ games $\mathcal{P}$-*positions* and $\mathcal{N}$-*positions* respectively.

A game G is written as $\{\mathcal{G}^L \mid \mathcal{G}^R\}$ where $\mathcal{G}^L$ is the list of Left options—positions that Left can reach in one move—and $\mathcal{G}^R$ is the list of Right options. If G and H are combinatorial games, then the *disjunctive sum* is denoted $G + H$ and is the game $\{\mathcal{G}^L + H, G + \mathcal{H}^L \mid \mathcal{G}^R + H, G + \mathcal{H}^R\}$, i.e. in which either player may play in one of the games but not both simultaneously. The game $-G$ is $\{-\mathcal{G}^R \mid -\mathcal{G}^L\}$, that is, the game obtained by reversing the roles of Left and Right. Moreover, we adopt the convention that $G + (-H) = G - H$. There is a defined 'equality' of games which induces an equivalence relation:

$$G = H \textit{ if } G - H \textit{ is a second player win.}$$

This turns out to be an important tool—to prove two games are equal, simply show that the second player can win the *difference* game. The game $\{\mid\}$ in which neither player has a move is called 0. *Any $\mathcal{P}$-position is equivalent to 0.* There is also a partial order: $G > H$ if $G - H$ is a Left win regardless of who goes first, $G < H$ if $G - H$ is a Right win regardless of who goes first. As an advantage to Left is regarded as positive and an advantage to Right is regarded as negative, we have:

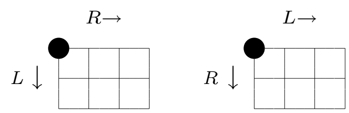
$$\begin{aligned} G > H &\quad \text{if} \quad G - H > 0 \\ G < H &\quad \text{if} \quad G - H < 0 \end{aligned}$$

Lastly, we say that G *is fuzzy with* H if $G - H$ is a first player win, ie:

$$G || H \textit{ if } G - H \textit{ is a first player win.}$$

Returning to our discussion of DIRECTIONAL PHUTBALL, note that the game $-P(m, n)$ is equivalent to the game $P(m, n)$ but with the roles of Left and Right reversed. That is, the game $-P(m, n)$ is the game $P(m, n)$ but with Right jumping vertically (instead of horizontally) and Left jumping horizontally (instead of vertically). It follows that $-P(m, n) = P(n, m)$.

The following example displays $P(3, 4) - P(3, 4)$:



P(3,4) -P(3,4)

From our definitions, it should follow that $P(3,4) - P(3,4) = 0$ since $P(3,4) = P(3,4)$. Recalling our definition of disjuctive sums and the fact that

$$P(3,4) - P(3,4) = P(3,4) + [-P(3,4)]$$

it is easily seen that the second player will always win by imitating the first player's moves in the reflected board. The winner, it is to be remembered, is the always the last player to move. This is a classic combinatorial game strategy called *Tweedledum-Tweedledee.* [2]

In this paper, we begin an analysis of DIRECTIONAL PHUTBALL, restricting considerations to $P(2,n)$. Section 3 gives a detailed analysis of $P(2,n)$ upto determining the poset of the values of the games. In the next Section, we present some values of DIRECTIONAL PHUTBALL games to demonstrate why the partial order is needed to approximate these values. Section 2 also contains some observations that are useful in the proofs contained in Section 3.

2. **Values and Observations.** In general, an equivalence class is referred to by an associated *value*. We have already encountered the value 0 which denotes the game $\{\,|\,\}$ where neither player has a move. As another example, the $\{0\,|\,\}$ in which Left has one move and Right has none, is called 1 and the game $\{\,|\,0\}$ in which Right has one move and Left has none is called -1. Notice that this adheres to our definitions above: a Left advantage is positive and a Right advantage is negative. For more details on the recursively defined values of games, see [1].

PHUTBALL and DIRECTIONAL PHUTBALL are *all-small* games, that is, games in which either both players have a move or neither player has a move (i.e. the game is over). The values of such games are infinitesimals, but only two are required for this paper. The first arise from the game of NIM played with one heap of counters (see the cited texts or search for 'the game of nim' to find out the strategy when playing with more than one heap). The rules of the game are as follows: on his or her turn, a player may remove as many of the counters as he or she wishes (possibly all of them) and, as is convention, the last player to remove counters wins. A heap with one counter is therefore the game $\{0 \mid 0\}$ which is defined as $*$; with 2 counters $\{0, * \mid 0, *\} = *2$. Recursively, a heap with $n+1$ counters is $\{0, *, *2, \ldots, *n \mid 0, *, *2, \ldots, *n\} = *(n+1)$. Note that $*n + *n = 0$ for any n.

The second set of values required for this paper are defined recursively by the game $\{0 \mid *\}$ and its negative. The game $\{0 \mid *\}$ is called $\uparrow$ and is a positive game since Left can win going first or second. Similarly, its negative $\{* \mid 0\}$ is called $\downarrow$.

One final note; letting $n \cdot G$ be the sum of n copies of G, it can be shown that $(n+1) \cdot (\uparrow + *) + * = \{0 \mid n \cdot (\uparrow + *) + *\}$. The basic fact to be gleaned from this recursive definition is that Right can only move to $n \cdot (\uparrow + *) + *$ and so requires essentially n moves to finish the game. On the other hand, Left can win the game immediately, even after letting Right have the first $n-1$ moves, by moving to 0.

Table 1 displays the values for small games of $P(n,m)$ in which $a := \{0, *2 \mid {\downarrow}{*3}\}$, $b := \{{\uparrow}{*3} \mid *2, a\}$, $c := \{\uparrow \mid *, \{0, * \mid \downarrow, *\}\}$, and
$d := \{\{0 \mid b\} \,|||\, \{b \,\|\, \{a, 0 \mid \{a, {\downarrow}{*3} \mid 0\}\}, \{b \mid 0, *\}, b\}\}$. These values were found using CG suite [6]

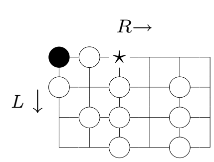
	n				
m	2	3	4	5	6
2	$*2$	$*2$	b	b	d
3	$*2$	$*$	b	c	d
4	$-b$	$-b$	$*2$		

TABLE 1. Values for the game $P(n,m)$.

These values are too complicated to remember while playing without a cheat sheet. Values for the games $P(2,7)$, $P(2,8)$, $P(2,9)$ and $P(2,10)$ were also found, they are unfortunately too awkward to be displayed here (the value for $P(2,8)$ is two lines long, $P(2,10)$ is 57 lines long). It is both surprising and unsurprising that DIRECTIONAL PHUTBALL has such complex values; surprising because of the seeming simplicity of the game, unsurprising because the values of all-small games are always infintesimals. Notice that $P(2,2) = P(2,3)$ and $P(2,4) = P(2,5)$; we will see in section 3 that indeed $P(2,2n) = P(2,2n+1)$. Similarly, $*2 < b < d$ so $P(2,2) < P(2,4) < P(2,6)$; the conjecture that $P(2,2n) < P(2,2n+2)$ will also be shown to be true. Unfortunately, due to the complicated nature of the values, we must rely on alternative methods (atomic weight and outcome classes) to determine this partial order among the games. As such, we need to make several initial observations that will be useful for the proofs in Section 3.

Observation 1. *Poultry (an off-parity placement) is not always a bad move.*

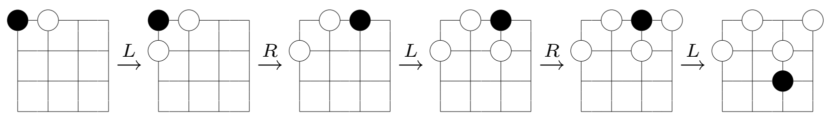
As in the case of 1-dimensional PHUTBALL, if both players have not played well, it is possible that the *only* winning move is an off-parity one. For instance, consider the following position:



And suppose it is Right's turn. We see upon close inspection that Right's only winning move is the placement of a man on $\star$. We will see in the case of sums of DIRECTIONAL PHUTBALL games that an off-parity placement can be a good move under both poor and intelligent play.

Observation 2. *In a game with only one component, the best strategy for Left and Right, playing either first or second, is to place a man adjacent to the ball in the direction that he or she is jumping then jump that man (if possible) on his or her next turn.*

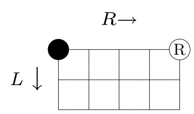
That is, under optimal play, the game would progress as follows (Right playing first):



The ball moving along the diagonal of the board is clearly optimal for both players. It may be seen by inspection that if a player deviates from this progression, the ball will skew in favour of his opponents' goal line. We may then conclude that the player who must travel the shorter length of the board wins playing first or second - if the board is a square, the first player wins.

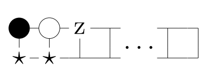
Observation 3. *No game of* DIRECTIONAL PHUTBALL *has a value of zero, that is, there are no board configurations that are Previous player wins.*

A short proof of this fact is as follows: suppose Right can win $P'(n,m)$ playing second. We wish to show that he may also win playing first. Let Right place his first man in the upper right hand corner of the board.



Right now plays his second player winning strategy, the only difference being that Right cannot jump onto the intersection occupied by Ⓡ, nor can he place a man on the intersection occupied by Ⓡ. The first consideration is of no consequence, since jumping over Ⓡ is equivalent to jumping into the intersection occupied by Ⓡ (he wins either way). The second consideration is easily overcome by the fact that if Right's second player strategy requires him to place a stone on Ⓡ, he may simply place a stone in the closest empty intersection to Ⓡ. If there are no empty spots to place a man, then Right can jump over the goal line and win. Thus Right wins playing first.

One further observation: Consider the game $P_1(2,n)$ below, where Right has played first to ○:



If Left plays anywhere other than z and the stared intersections, Right can respond *as if* he were playing second in $P(2,n-2)$ since Right can make all the same moves as are available to him in $P(2,n-2)$. If Left plays in either $\star$, Right can jump to z, and the game becomes $P(2,n-2)$ with Left to play first. The only catch is if Left plays in z. So:

Observation 4.

$$P_1(2,n) \sim P(2,n-2), \qquad \textit{for Right, unless Left plays in } \mathrm{z}$$

Where $\sim$ means "is equivalent to" with the proviso given above.

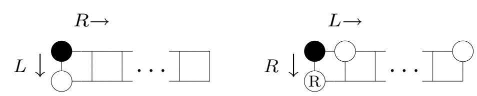
3. $2 \times n$ **Directional Phutball.** We have seen that the strategy and outcome for a single game of DIRECTIONAL PHUTBALL is fairly trivial. However, playing the *sum* of DIRECTIONAL PHUTBALL games is far from straightforward. This can be seen from the complicated nature of the table of values. We now begin our assessment of $2 \times n$ DIRECTIONAL PHUTBALL with the intention of finding the outcome of the sums of such games, and hence their partial order.

Because we will be dealing mainly with games of the form $P(2,n) - P(2,k)$ in the coming sections, we begin with a simple Lemma (recall that $P'(m,n)$ represents the game $P(m,n)$ in progress):

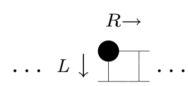
Lemma 1. *Given the game $P'(2,n) - P'(2,k)$ in progress, if Left can jump to 0 in both $P'(2,n)$ and $-P'(2,k)$, and Right cannot jump to 0 in either, then $P'(2,n) - P'(2,k) > 0$.*

Proof. Suppose Left can jump to 0 in both $P'(2,n)$ and $-P'(2,k)$, Right cannot jump to 0 in either. If Left plays first, she jumps to 0 in $P'(2,n)$. Right cannot jump in $-P'(2,k)$, so can only place a man. Left jumps to win on her next turn.

Now suppose it is Right's turn first. The *best* Right can do is to make a move such that he will be able to jump to zero in either $P'(2,n)$ or $-P'(2,k)$ (If he cannot do this, then Left wins by the reasoning above).
Suppose Right plays Ⓡ in $-P'(2,k)$:



Left then jumps to zero in $-P'(2,k)$, and Right must now play in $P'(2,n)$. If Right does not jump, Left will clearly win on her next turn. Since Right cannot jump to 0 in $P'(2,n)$, we know the best she can do is to jump to some position of the form:



With any possible array of men to the left and right. Left wins since this is, at best for Right, a next player win. If Right attempts to stall his jump along his series of men in $P'(2,n)$ (i.e. only jump part way) Left simply puts a shot in below the ball, forcing Right to continue her jump until he has reached the position shown above. Thus, Ⓡ is a bad move for Right. If Right has some move in $P'(2,n)$ that enables him to jump to zero in $P'(2,n)$, Left jumps to zero in $P'(2,n)$ and Right loses playing first in $-P'(2,k)$. □

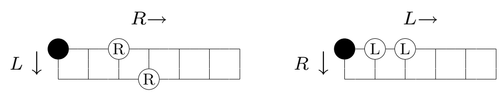
A noteworthy observation stemming from Lemma 1 is as follows:

Corollary 1. *If Left can jump to zero in both $P'(2,n)$ and $-P'(2,k)$, Right can jump to 0 in one of $P'(2,n)$ or $-P'(2,k)$ and it is Left's turn, Left wins.*

Proof. The proof can be seen directly from the proof of Lemma 1. □

We will refer to a position in which a player can jump to zero in both $P'(2,n)$ and $-P'(2,k)$ as a *Knock Out*, or *KO* position. Notice that both players can simultaneously be in KO positions, in which case the game is at a *standstill* and the outcome of the game is unclear[1] .

We define an *efficient* move to be a placement that that decreases the number of placements necessary to achieve a KO position by 1. In the following example, Left as made two efficient moves and Right has made two inefficient moves in the game $P(2,7) - P(2,6)$ (inspect carefully; notice both Right and Left require 4 moves to reach a KO position):



Notice that a jump (that is not to zero) is not an efficient move since it does not decrease the number of moves requires to attain a KO position.

Corollary 2. *In some game of* DIRECTIONAL PHUTBALL $P'(2,n) - P'(2,k)$*, if the next player requires* k *efficient moves to achieve a KO position, and the previous player requires* $k + j$ *where* $j \geq 1$*, then the next player will win.*

Proof. The next player will win by the following strategy: depending on whether the next player is Right or Left, he or she plays efficiently in $P'(2,n)$ or $-P'(2,k)$ respectively, then places his or her last move beneath the ball in $-P'(2,k)$ or $P'(2,n)$ respectively. Let the game progress until the next player achieves his KO position. Now it is the previous players' turn; notice that if the previous player has played efficiently, he or she will require exactly $j + 1$ moves to achieve a KO position. Otherwise, he or she will require $j+k+1$ where k is the number of inefficient moves he or she has made. Therefore, the previous player requires *at least* $j + 1$ where $j \geq 1$ more moves to achieve his KO position. The best he or she can do is get into a position such that he or she can jump to zero in one of $P'(2,n)$ or $-P'(2,k)$. The next player wins by the comments following Lemma 1. If the previous player had, at some point, jumped to zero in one of $P'(2,n)$ or $-P'(2,k)$, the next player wins since, at worst, it will require the next player 2 moves to jump to zero in the board still in play, but the previous player, at best, requires 4. □

3.1. **Atomic Weight or Comparison with Multiples of $\uparrow$.** The *Atomic Weight* of a game G is k if $2 \cdot \downarrow + * < G - k \cdot \uparrow < 2 \cdot \uparrow + *$. It corresponds to the multiple of $\uparrow$s that closely approximates the game (atomic weight is in fact a homomorphism from the class of games to itself, so the appropriate multiple doesn't have to be an integer, in fact, not even a number). For DIRECTIONAL PHUTBALL, it appears as if the atomic weights are all integers. This then allows the definition of $k \cdot \uparrow$ given in the first section to be the guide; if the atomic weight is the positive integer k, then Right requires k moves before he can win the game, whereas Left can win the game after Right has made $m < k$ moves. For example, the value of $P(2,8)$ is two lines long, however, its atomic weight is 2 meaning that it has a value of $2 \cdot \uparrow$ plus some other, even smaller, infinitesimals.

The atomic weights for some values of $P(n,m)$ are shown in Table 2. Again, these are provided by CGsuite [6].

[1]The last player to have placed a stone, the parity of the number of empty tiles remaining, and the partial jumps available to both players are all factors in determining the outcome of a standstill.

m	n = 2	3	4	5	6	7	8
2	0	0	0	0	1	1	2
3	0	0	0	0	1		
4	0	0	0	0			

TABLE 2. Atomic weights for the game $P(n, m)$.

The following Theorem does not provide specific atomic weights, but bounds all games of the form $P(2, m)$ in terms of $\uparrow$s and $*$s:

Theorem 1. *For the game $P(2, 2n+1)$ with $n \geq 3$ we have:*

1. $P(2, 2n+1) \| (n-2)\uparrow + (n-2)*$
2. $(n-3)\uparrow + (n-3)* < P(2, 2n+1) < (n-1)\uparrow + (n-1)*$.

Proof. 1. To show that $P(2, 2n+1) \| (n-2)\uparrow + (n-2)*$, we must show that $P(2, 2n+1) + (n-2)\downarrow + (n-2)*$ is a first player win. Right requires n moves to jump to zero in $P(2, 2n+1)$. Left must play $n-1$ moves to finish $(n-2)\uparrow + (n-2)*$, and requires one shot beneath the ball in $P(2, 2n+1)$. Thus Left needs to play $n-1+1 = n$ moves in total in order to jump to zero in $P(2, 2n+1) + (n-2)\downarrow + (n-2)*$. The first player wins.

2. To show that $(n-3)\uparrow + (n-3)* < P(2, 2n+1)$, we must show that $0 < (n-3)\downarrow + (n-3)* + P(2, 2n+1)$. Right may finish, or jump to zero in $(n-3)\downarrow + (n-3)*$ at any time, but Left requires $n-3$ moves in before he would be able to do so. Thus, in total, Left requires $n-3+1 = n-2$ moves (one extra move for beneath the ball in $P(2, 2n+1)$) before he reaches a KO position, whereas Right requires n moves. Left wins by Corollary 2.

   To show that $P(2, 2n+1) < (n-2)\uparrow + (n-2)*$, we must show that $P(2, 2n+1) + (n-1)\downarrow + (n-1)* < 0$. By the same argument, Right requires n moves to reach a KO position, and Left requires $n-2+1 = n-1$. Right wins by Corollary 2.

□

A noteworthy Corollary of Theorem 1 is as follows:

Corollary 3. *In the game of* DIRECTIONAL PHUTBALL*,*

$$P(2, 2n+5) > P(2, 2n+1) \text{ for } n > 1$$

Proof. From Theorem 1 we have:

$$P(2, 2n+1) < (n-1)\uparrow + (n-1)* = ((n+2)-3)\uparrow + ((n+2)-3)* < P(2, 2n+5)$$

□

Furthermore, it is readily seen that $P(2, 2n+5+i) > P(2, 2n+1)$ for $n > 1$ and $i > 1$: Since $P(2, 2n+5) - P(2, 2n+1) > 0$ by the fact that Right requires too many efficient moves to win, he will require *at least* as many in the game $P(2, 2n+5+i) - P(2, 2n+1)$, thus loses such a game playing either first or second.

We may now construct our first approximation of the partial order of $2 \times n$ DIRECTIONAL PHUTBALL games in a Hasse diagram. In such a diagram, two games are connected by a line if one is greater than the other; the greater game above the smaller one. When two games are incomparable, they are placed on the same horizontal but not connected by a line.

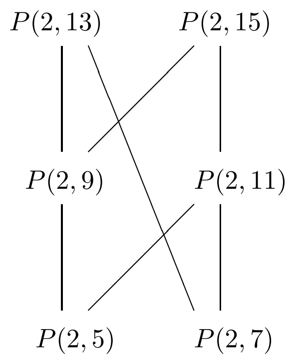
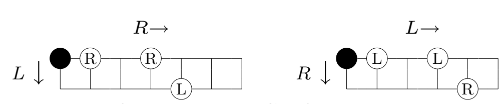


FIGURE 2. Approximation of the partial order of games $P(2,n)$ where $n = \{5, 7 \ldots 15\}$.

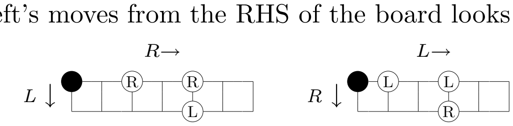
The above approximation leaves room for improvement. The order of even board lengths is yet unknown - we will see in the next section that in fact $P(2, 2n+1) = P(2, 2n)$. The relation between boards that differ by two or three columns is unclear and is simply estimated in Figure 2. Direct comparisons will be used to finalize a partial order.

3.2. **Improved Partial Order of $P(2,n)$ Games.** As mentioned above, $P(2, 2n+1) - P(2, 2n)$ is in fact a second player win, that is, $P(2, 2n+1) = P(2, 2n)$. The proof is somewhat technical, so a short synopsis will be given here. Notice there is *nearly* a 1-1 correspondence between the two boards, except for 1 extra column (or extra vertical line) in $P(2, 2n+1)$. The second player's goal is to eliminate this extra column from play (either by placing men in it, or leaving it blank), so that there is an exact 1-1 correspondence between the two boards. The second player will then win by Tweedledum-Tweedledee, that is, by imitating the first player's moves. However, the second player must eliminate the appropriate column by imitating the first player's moves in such a way that once the first player makes the location of the extra column apparent (generally by making an off-parity move in one of the boards) that column may be eliminated, and the two boards will be equal. The First player may also make a losing move to the effect that the Second player need only play efficiently to win.

In the following proof, I will refer to *imitation from the LHS* or *RHS* of the board. For instance, in the following example, Right playing second (R) is imitating Left's moves (L) from the LHS of the board:



Right imitating Left's moves from the RHS of the board looks like this:

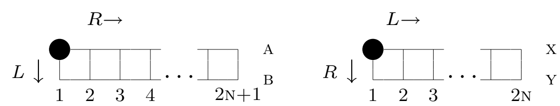


We now begin our proof:

Theorem 2. *In the game of* DIRECTIONAL PHUTBALL*,*

$$P(2, 2n + 1) = P(2, 2n) \text{ for } n \geq 1$$

Proof. First, we label the boards of the game $P(2, 2n+1) - P(2, 2n)$ as follows:



We will, for example, denote the tile in the *Ath* row and *ith* column by (A, i).

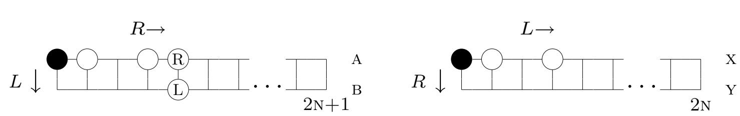
Left wins playing second with the following strategy: If Right places a stone that is **not** in:

1. Row X of $-P(2, 2n)$ such that there are no men in any of the columns to the right of that stone,
2. (A, i) or (B, i) where i is odd, there is a man in $(A, i-1)$ and there are no men in any of the columns to the right of i,
3. (A, i) where i is odd, there is *no* man in $(A, i-1)$ and there are no men in any of the columns to the right of i,
4. (Y, i) or (B, i) for any i (except for the case in 2.) and there are no men in any of the columns to the right of i

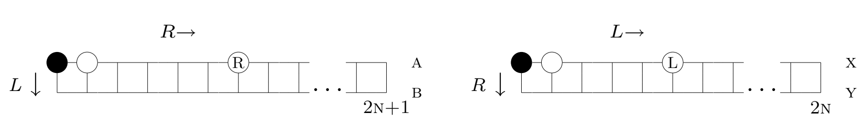
Then Left imitates Right's move from the LHS of the board — this will *always* be possible since jumping off the end of the board in $-P(2, 2n)$ is equivalent to jumping into the last column of $P(2, 2n+1)$ as long as men are only placed on the oddish tiles.

If Right plays 1, Left responds by playing an efficient move. Left now requires k moves to be in a KO position, whereas Right requires $k + 3$, and it is Right's turn. Right loses by Corollary 2.

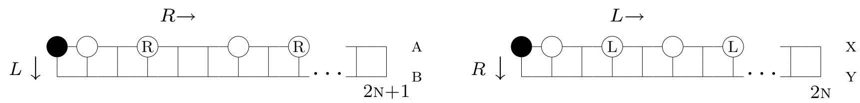
If Right plays 2, Left plays (B, i) or (A, i) respectively. Left has eliminated column i, and the boards are equal with Left playing second, so Left wins. For example, if Right plays Ⓡ then Left plays Ⓛ:



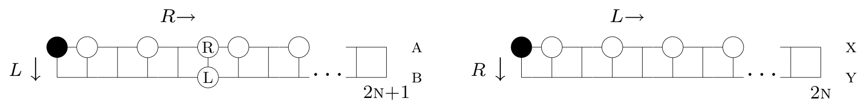
If Right plays 3, Left plays $(X, i-1)$, mimicking Right's move as if Right has already made a play as in case 2.:



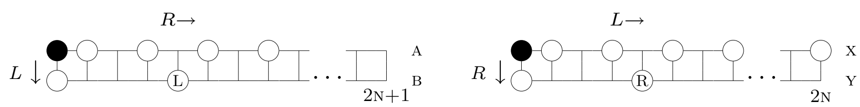
Left now mimics all of Right's moves to the right of (A, i) from the RHS of the board, and all moves to the left of (A, i) from the LHS of the board.



Eventually, Right is forced to play either a man in $(A, i-1)$ or make an off parity move in row A to the left of i, in which case Left does as in case 1. and eliminates the column:



If Right plays 4, Left responds by playing an efficient move. Left now requires k moves to achieve a KO position, whereas Right requires $k+1$. Right must therefore play only efficient moves from here on in (strictly oddish moves), otherwise, he will be at least 2 efficient moves behind Left, and Left will win by Corollary 2. Eventually, Left plays to a KO position, Right plays to a KO position, and it is Left's turn. Left now plays $(\cdot, i)$ (Ⓛ) where Right's case 4. move was of the form $(\cdot', i)$ (Ⓡ) and $\{\cdot, \cdot'\} = \{Y, B\}$.



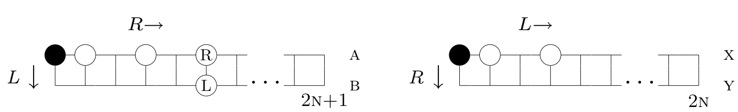
Left now considers column $2n+1$ in $P(2, 2n+1)$ as eliminated from play; Left mimics Right's moves as if column $2n+1$ does not exist. If Right places a man in $(\cdot, 2n+1)$, Left places a man in $(\cdot', 2n+1)$ where $\{\cdot, \cdot'\} = \{A, B\}$.

Right wins playing $P(2, 2n+1) - P(2, 2n)$ second with the following strategy: If Left places a stone that is **not** in:

1. An empty column in $P(2, 2n+1)$ such that there are no men in any of the columns to the right of that move,
2. (X, i) where i is odd and there are no men in any of the columns to the right of i,
3. (Y, j) for any j, there is no stone in (X, j) and there are no men in any of the columns to the right of j,

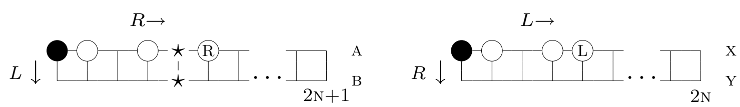
then Right imitates Left's moves from the LHS of the board- this will *always* be possible since jumping off the end of the board in $-P(2, 2n)$ is equivalent to jumping into the last column of $P(2, 2n+1)$ as long as men are only placed on the oddish tiles.

If Left plays 1, then Right moves in the same column. That is, if Left plays $(\cdot, i)$ then Right plays $(\cdot', i)$ where $\{\cdot, \cdot'\} = \{A, B\}$. Right has now eliminated the column from play, and although Right may be able to make a move that Left cannot, Left is not able to make any moves that Right cannot. For example, if Left plays Ⓛ, Right plays Ⓡ:



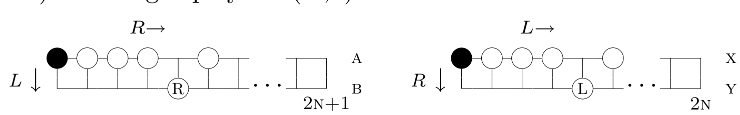
Thus the boards (from Right's point of view) are equal with Left playing second; Right wins by Tweedledum-Tweedledee.

If Left plays 2, Right plays $(A, i+1)$. Right now considers the column i (the starred column in the example below) as eliminated from play and mimics Lefts remaining moves accordingly unless Left places a man in $(\cdot, i)$ in which case Right puts a man in $(\cdot', i)$ where $\{\cdot, \cdot'\} = \{A, B\}$.

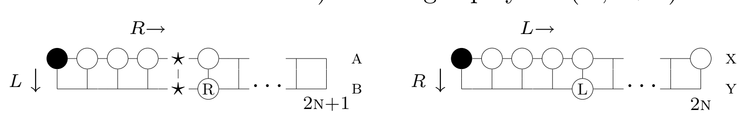


In any case, the boards are equal (from Right's point of view) and Right wins playing second.

If Left plays 3, Right plays an efficient move, and furthermore, plays an efficient strategy from now on wherein he plays only on the oddish intersections in A of $P(2, 2n+1)$ and places his last stone beneath the ball in $-P(2, 2n)$. ILeft must now play: since she cannot use the man in (Y, j) to her advantage in any way, it will take her $k+1$ moves to be in a KO position, whereas it will take Right k moves only. Thus, Left must play efficiently from now on, otherwise she will lose by Corollary 2 (notice playing efficiently for Left means she may have *at most* 1 even clump of men side by side in row X). Eventually, Right will be in a KO position, Left will be in a KO position and it is Right's turn. If Left's move in (Y, i) (L in the example below) was on parity (i.e. i is odd) and there is no man in (X, i) (notice that no man in (X, i) implies that that if there is a clump of even men in row X, it is to the right of i) then Right plays in (B, i):



If there is a man in (X, i) (notice that a man in (X, i) implies that there is a clump of even men in row X to the left of i) then Right plays in $(B, i+1)$:



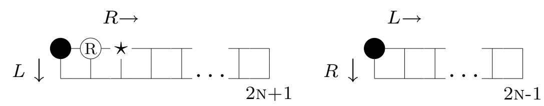
Right considers the column r containing the rightmost man in the even clump in row X as eliminated in $P(2, 2n+1)$ (the starred column in the example above); if Left plays $(\cdot, r)$ then Right plays $(\cdot', r)$ where $\{\cdot, \cdot'\} = \{A, B\}$. Otherwise, Right mimics Left's remaining moves accordingly, that is, as if column r in $P(2, 2n+1)$ had been eliminated. If there is no even clump of men in row X, Right considers column $2n+1$ in $P(2, 2n+1)$ as eliminated.

□

Theorem 3. *In the game of* DIRECTIONAL PHUTBALL,

$$P(2, 2n+1) \| P(2, 2n-1) \textit{ for } n > 2$$

Proof. Ⓡ is a winning move for Right playing first in $P(2, 2n+1) - P(2, 2n-1)$:



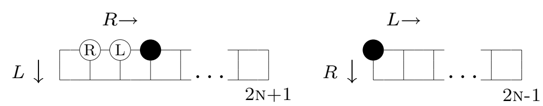
Letting $P_1(2, 2n+1)$ be the game $P(2, 2n+1)$ in progress shown above, Since

$$P_1(2, 2n+1) \sim P(2, 2n-1)$$

for Right unless Left places a man in $\star$ (by Observation 4), we have:

$$P_1(2, 2n+1) - P(2, 2n-1) \sim P(2, 2n-1) - P(2, 2n-1) = 0$$

with Right to play second. Thus Left's only possible good move is to place a man on $\star$. Right responds by jumping:



The game becomes:

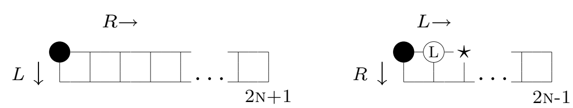
$$P(2, 2n-2) - P(2, 2n-1)$$

and from Theorem 2 we have:

$$-[P(2, 2n-1) - P(2, 2n-2)] = 0$$

with Right to play second. Thus Right wins playing first.

Ⓛ is a winning move for Left playing first in $P(2, 2n+1) - P(2, 2n-1)$:



Letting $P_2(2, 2n-1)$ be the game $P(2, 2n-1)$ in progress shown above, since

$$P_2(2, 2n-1) \sim P(2, 2n-3)$$

for Left unless Right plays $\star$ (by Observation 4) we have:

$$P(2, 2n+1) - P_2(2, 2n-1) \sim P(2, 2n+1) - P(2, 2n-3)$$

But from Corollary 3:

$$P(2, 2n+1) - P(2, 2n-3) > 0$$

So Right's only possible good move is to place a man on $\star$. Left now jumps in $P_2(2, 2n-1)$ and the game becomes:

$$P(2, 2n+1) - P(2, 2n-4) > 0$$

Right is worse off. Left wins playing first. □

Theorem 4. *In the game of* DIRECTIONAL PHUTBALL*,*
1. $P(2, 2n) \| P(2, 2n-1)$ *for* $n > 2$
2. $P(2, 2n+2) \| P(2, 2n)$ *for* $n > 2$

Proof. 1. From Theorem 2 and Theorem 3 we have:

$$P(2, 2n) = P(2, 2n+1) \| P(2, 2n-1)$$

2. From Theorem 2 and Theorem 3 we have:

$$P(2, 2n + 2) = P(2, 2n + 3) \| P(2, 2n + 1) = P(2, 2n)$$

$\square$

We are now prepared to build the complete partial order of $P(2, n)$ games. Figure 3 displays this partial order for $n = \{2 \ldots 17\}$ in a Hasse diagram.

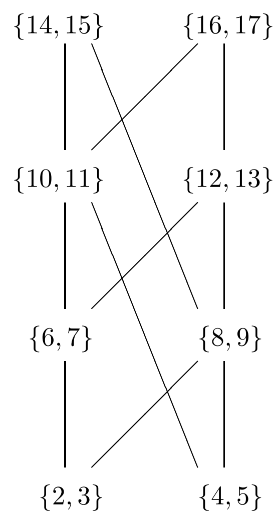
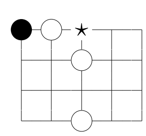


FIGURE 3. The partial order of games $P(2, n)$ where $n = \{2, \ldots, 17\}$.

4. **Open Questions.** Consideration of $P(m, n)$ for $m \geq 3$ is an obvious route for further analysis. It might also be fruitful to consider the less restricted game of DIRECTIONAL PHUTBALL in which both players may jump in either direction, that is, vertically or horizontally, but their goal lines remain on the rightmost and bottommost edges of the board. That is, Right aims is to jump either onto or over the rightmost goal line, and Left aims to jump either onto or over the bottommost goal line. This is a more realistic game, and resembles PHUTBALL more closely while still maintaining the (desired?) non-loopiness property. More interesting strategies may also be applied; for instance, Right may avert Left winning in the following game by placing a man on $\star$:



This is a PHUTBALL strategy called a *tackle* [2]. Tackles do not occur in DIRECTIONAL PHUTBALL.

Acknowledgments. I would like to extend my deepest gratitude to Richard Nowakowski, to whom I am indebted not only for his technical help with this paper, but also for the guidance, inspiration and patience which made this project possible. I would also like to thank David Wolfe, whose macros, available at `http://homepages.gac.edu/~wolfe/lessonsinplay/latex-macros/` were used to create the diagrams that appear in this paper. Finally, thanks to Adam Brown for his help and advice.

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