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SOLVING SEXTIC EQUATIONS

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Abstract. This paper presents a novel decomposition method to solve various types of solvable sextic equations.

1. **Introduction.** The works of Abel (1826) and Galois (1832) have shown that the general polynomial equations of degree higher than the fourth cannot be solved in radicals [1]. While Abel published the proof of impossibility of solving these equations (Abel's impossibility theorem), Galois gave a more rigorous proof using the group theory. This does not mean that there is no algebraic solution to the general polynomial equations of degree five and above [2]. In fact these equations are solved algebraically by employing symbolic coefficients: the general quintic is solved by using the Bring radicals, while the general sextic can be solved in terms of Kampe de Feriet functions [3].

In this paper, we describe a method to decompose the given sextic equation (sixth-degree polynomial equation) into two cubic polynomials as factors. The cubic polynomials are then equated to zero and solved to obtain the six roots of the sextic equation in radicals. The salient feature of the sextic solved in this manner is that, the sum of its three roots is equal to the sum of its remaining three roots. The condition required to be satisfied by the coefficients of such solvable sextic is derived. A numerical example is solved in the last section using the method presented.

2. **Decomposition of sextic equation.** Let the sextic equation whose solution is sought be:

$$x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0 \tag{1}$$

where a_0, a_1, a_2, a_3, a_4 and a_5 are the real coefficients in the above equation. Consider another sextic equation as shown below:

$$(x^3 + b_2x^2 + b_1x + b_0)^2 - (c_2x^2 + c_1x + c_0)^2 = 0 \tag{2}$$

where b_0, b_1, b_2, and c_0, c_1, c_2, are the unknown coefficients in the constituent cubic and quadratic polynomials respectively, in the above equation. Notice that the sextic equation (2) can be easily decomposed into two factors as shown below.

$$[x^3+(b_2-c_2)x^2+(b_1-c_1)x+b_0-c_0][x^3+(b_2+c_2)x^2+(b_1+c_1)x+b_0+c_0] = 0 \tag{3}$$

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Therefore, if the given sextic equation (1) can be represented in the form of (2), then it can be factored into two cubic polynomial factors as shown in (3), leading to its solution. These polynomial factors are equated to zero to obtain the following cubic equations.

$$
\begin{aligned}
&x^3+(b_2-c_2)x^2+(b_1-c_1)x+b_0-c_0=0\\
&x^3+(b_2+c_2)x^2+(b_1+c_1)x+b_0+c_0=0
\end{aligned}
\tag{4}
$$

The six roots of the given sextic equation (1) are then obtained by solving the above cubic equations. Thus in order to represent the given sextic equation (1) in the form of (2), the coefficients of (1) should be equal to the coefficients of (2). However the coefficients of (2) are not explicitly available. Therefore the sextic equation (2) is now expanded and rearranged in descending powers of x as shown below, to facilitate equating its coefficients with that of sextic equation (1).

$$
\begin{aligned}
x^6+2b_2x^5+(b_2^2+2b_1-c_2^2)x^4+2(b_0+b_1b_2-c_1c_2)x^3\\
+[b_1^2+2b_0b_2-(c_1^2+2c_0c_2)]x^2+2(b_0b_1-c_0c_1)x+b_0^2-c_0^2=0
\end{aligned}
\tag{5}
$$

Equating the coefficients of (5) with the coefficients of given sextic equation (1), results into following six equations.

$$2b_2=a_5 \tag{6}$$

$$b_2^2+2b_1-c_2^2=a_4 \tag{7}$$

$$2(b_0+b_1b_2-c_1c_2)=a_3 \tag{8}$$

$$b_1^2+2b_0b_2-(c_1^2+2c_0c_2)=a_2 \tag{9}$$

$$2(b_0b_1-c_0c_1)=a_1 \tag{10}$$

$$b_0^2-c_0^2=a_0 \tag{11}$$

Notice that even though there are six equations [(6) to (11)] to determine the six unknowns (b_0, b_1, b_2, c_0, c_1, and c_2), all the unknowns cannot be determined since solution to the general sextic equation is not possible in radicals. Since our aim is to solve sextic equation in radicals, we introduce one more equation called supplementary equation to determine all the six unknowns. The supplementary equation introduced will decide the type of solvable sextic. Let the supplementary equation introduced be as follows.

$$c_2=0 \tag{12}$$

From (6) b_2 is evaluated as:

$$b_2=a_5/2 \tag{13}$$

Using (12) and (13) in equation (7), b_1 is found out as:

$$b_1=(a_4/2)-(a_5^2/8) \tag{14}$$

Using (12), (13), and (14), the values of c_2, b_2, and b_1 are substituted in equation (8) to evaluate b_0 as:

$$b_0=(a_3/2)+(a_5^3/16)-(a_4a_5/4) \tag{15}$$

Using (12), (13), (14), and (15) respectively, the values of c_2, b_2, b_1, and b_0 are substituted in equation (9), to obtain the value of c_1^2 as:

$$c_1^2=[(5a_5^4/64)-(3a_4a_5^2/8)+(a_4^2/4)+(a_3a_5/2)-a_2] \tag{16}$$

From the above expression, we notice that c_1 has two values, c_{11} and c_{12}, as indicated below.

$$\begin{aligned} c_{11} &= a_6 \\ c_{12} &= -a_6 \end{aligned} \tag{17}$$

where a_6 is given by:

$$a_6 = \sqrt{(5a_5^4/64) - (3a_4a_5^2/8) + (a_4^2/4) + (a_3a_5/2) - a_2} \tag{18}$$

Observing the above expression for a_6, the curious reader may wonder whether the proposed method works when a_6 becomes imaginary [i.e., when the term under the square-root sign in (18) becomes negative]; yes, it does work as demonstrated by the numerical example given in the last section. The only condition on a_6 is that it should not be zero (see the next paragraph).

Consider equation (10). Substitute the values of c_2, b_2, b_1, b_0, and c_1 using (12), (13), (14), (15), and (17) respectively in (10), to determine c_0. Since c_1 has two values, c_{11} and c_{12}; c_0 also has corresponding two values, c_{01} and c_{02}, as shown below (for $a_6 \neq 0$).

$$\begin{aligned} c_{01} &= a_7/a_6 \\ c_{02} &= -a_7/a_6 \end{aligned} \tag{19}$$

where a_7 is given by:

$$a_7 = (a_3a_4/4) + (a_4a_5^3/16) - (a_4^2a_5/8) - (a_3a_5^2/16) - (a_5^5/128) - (a_1/2) \tag{20}$$

We have determined all the unknowns, and substituting their values in the cubic equation set (4), we get the following pair of cubic equations.

$$\begin{aligned} x^3 + b_2x^2 + (b_1 - a_6)x + b_0 - (a_7/a_6) &= 0 \\ x^3 + b_2x^2 + (b_1 + a_6)x + b_0 + (a_7/a_6) &= 0 \end{aligned} \tag{21}$$

Notice that, even though there are two values each for c_0 and c_1, both values will yield same pair of cubic equations. These equations are then solved to obtain all the six roots of the given sextic equation (1).

3. **Behavior of roots.** Let x_1, x_2, x_3, x_4, x_5, and x_6 be the roots of given sextic equation; let x_1, x_2, and x_3 be the roots of first cubic equation in (21), and x_4, x_5, and x_6 be the roots of second cubic equation. Notice that x^2 term in both the cubic equations is same, which means the sum of the roots, x_1, x_2, and x_3, is equal to the sum of the roots, x_4, x_5, and x_6; it then follows that this sum is equal to: $(-a_5/2)$. Thus the roots are related as shown below.

$$x_1 + x_2 + x_3 = x_4 + x_5 + x_6 \tag{22}$$

$$x_4 + x_5 + x_6 = (-a_5/2) \tag{23}$$

We note that the solvable sextic equation has one dependent root, and is determined from the remaining five roots through the relation (22).

4. **Condition for coefficients.** Since one of the roots of sextic equation (1) is a dependent root, one of the coefficients also will be a dependent coefficient, and it will be determined by the remaining coefficients. To derive this relation (among the coefficients) consider the equation (11), and substitute the values of b_0 and c_0 in (11) using (15) and (19) respectively. We obtain the following expression containing only the coefficients of (1).

$$a_0 = [(a_3/2) + (a_5^3/16) - (a_4 a_5/4)]^2 - (a_7/a_6)^2 \tag{24}$$

Expression (24) is the condition for the coefficients to satisfy, so that the given sextic equation (1) can be solved by the proposed technique. Notice that the coefficient, a_0, can be determined from the remaining coefficients using the relation (24).

5. **A note on the supplementary equation.** In section 2 we introduced one supplementary equation, $c_2 = 0$, to facilitate evaluation of unknowns. The supplementary equation chosen decides the type of solvable sextic equation. Notice that if we introduce, $c_0 = 0$, as supplementary equation, then the solvable sextic equation has its product of three roots (out of six roots) being equal to the product of its remaining three roots, as indicated below.

$$x_1 x_2 x_3 = x_4 x_5 x_6$$

The interested reader is invited to solve the sextic equation using the supplementary equation, $c_0 = 0$, and prove that the roots are related as mentioned above.

6. **Numerical example.** Let us solve the following sextic equation using the proposed method.

$$x^6 - 8x^5 + 32x^4 - 78x^3 + 121x^2 - 110x + 50 = 0$$

First step is to check whether the coefficients in the above sextic equation satisfy the condition stipulated by the expression (24), or not. Evaluating a_0 from the expression (24), we obtain $a_0 = 50$, and thus we note that this condition is met. Using (13), (14), (15), (18), and (20), b_2, b_1, b_0, c_1, and c_0, are evaluated as: $b_2 = -4$, $b_1 = 8$, $b_0 = -7$, $c_1 = i$, and $c_0 = i$ respectively; where $i = \sqrt{-1}$. Using these values in (4), the pair of cubic equations obtained is:

$$x^3 - 4x^2 + (8 + i)x - 7 + i = 0$$

$$x^3 - 4x^2 + (8 - i)x - 7 - i = 0$$

The roots of the first cubic equation in the above pair are determined from the well-known methods [4] as: $1 + i$, $2 + i$, and $1 - 2i$; and the roots of second cubic equation are evaluated as: $1 - i$, $2 - i$, and $1 + 2i$. Thus all the six roots of the sextic equation are found out.

7. **Conclusions.** A method to solve various types of solvable sextic equations is described. It is shown that, for one of such solvable sextic equations, the sum of its three roots is equal to the sum of its remaining three roots. The condition to be satisfied by the coefficients of such solvable sextic is derived.

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