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A BASIS FOR THE k-NORMALIZATION OF SEMIGROUPS

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Abstract. The depth of a term is an inductively defined measure of its complexity. For any natural number $k \geq 1$, an identity $s \approx t$ is said to be k-normal, with respect to the depth measurement, if either $s = t$ or both s and t have depth at least k. A variety V of algebras is said to be k-normal if all the identities satisfied by V are k-normal. For any variety V of algebras, the k-normalization of V is the variety defined by all the k-normal identities satisfied in V. This is the smallest k-normal variety to contain V.
A semigroup is an algebra with one binary operation which satisfies the associative law. Let Sem be the variety of all semigroups and let $N_k(Sem)$ be the k-normalization of Sem. The variety $N_k(Sem)$ is the equational class of algebras that satisfy all k-normal consequences of associativity. In this paper we produce a finite equational basis for $N_k(Sem)$, for $k \geq 3$.

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1. **Introduction.** The goal of this paper is to produce a finite equational basis for the varieties $N_k(Sem)$, where Sem is the variety of all semigroups and N_k is the k-normalization operator, for $k \geq 3$. In this section we provide a brief introduction to the study of universal algebra, defining algebras, varieties, identities, and bases. In Section 2 we define k-normal identities and varieties, and introduce the k-normalization operator N_k. Section 3 presents our main theorem, the basis for $N_k(Sem)$ for $k \geq 3$. This theorem is proved by induction on k, with some preliminary lemmas and the base case in Section 4 and the remainder of

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the proof in Section 5. A final appendix gives fuller details of all the calculations needed for the base case.

Universal algebra is the study of algebras and identities. An *algebra* is a set of objects with one or more operations defined on the set. The *type* of the algebra is a list of the arities of the operations being used. For example, a group can be regarded as an algebra of type (2), with a single binary operation. An *identity* of an algebra is an equation of two expressions which holds for all elements of the algebra. For example, a commutative group satisfies $f(x_1, x_2) \approx f(x_2, x_1)$, for every x_1 and x_2 in the group, where the symbol f denotes the binary operation of the group. Expressions such as $f(x_1, x_2)$ and $f(x_2, x_1)$ are called *terms* in the language of the type.

More formally, we let $\tau = (n_i)_{i \in I}$ be any type of algebras, with an operation symbol f_i of arity n_i for each $i \in I$. For each $n \geq 1$, we define the n-ary terms of type τ by the following inductive definition:

1. Each variable $x_1, x_2, \ldots, x_n$ is an n-ary term.
2. If $t_1, \ldots, t_{n_i}$ are n-ary terms and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term.

Any n-ary term for some $n \geq 1$ is then called a term. Any term t can be represented by a tree diagram, with the root labelled by the outermost operation symbol of t, each branch of the tree labelled by an operation symbol and each leaf of the tree labelled by a variable. For example, Figure 1 shows the term $t = f(g(x_1, f(x_2, x_3)), x_4)$, where f and g are binary operation symbols.

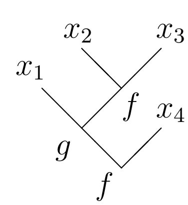


Figure 1: Tree Diagram for Term $t = f(g(x_1, f(x_2, x_3)), x_4)$

An *identity* of type τ is an expression of the form $s \approx t$ where s and t are terms of the type. An algebra $\mathcal{A}$ is said to *satisfy* an identity $s \approx t$ if s and t give equal results no matter what elements of $\mathcal{A}$ are substituted for the variables $x_1, x_2, \ldots$ in the terms. We can use the relationship of satisfaction (of an identity by an algebra) to go back and forth between collections of algebras and identities. For any class K of algebras of type τ and any set Σ of identities of type τ, we can consider the collection of all algebras which satisfy all the identities in Σ, called the *models* of Σ; and similarly the collection of all identities satisfied by all the algebras in K. This defines a pair of operators called Mod

and Id: for any classes Σ and K, $Mod\,\Sigma$ is the class of all algebras $\mathcal{A}$ of type τ which satisfy all the identities in Σ, and $Id\,K$ is the set of all identities $s \approx t$ of type τ which are satisfied by all algebras in K.
These operators Id and Mod satisfy certain nice properties which make them into what is called a *Galois correspondence*. In particular, the composition operators $Mod\ Id$ and $Id\ Mod$ are closure operators, on the sets of all algebras of type τ and all identities of type τ respectively. Much can then be said about the closed sets on either side of this correspondence. The closed collections of algebras, that is classes K for which $Mod\ IdK = K$, are called *equational classes* or *varieties*.
On the other side of the Galois correspondence, the closed sets of identities are called *equational theories*. An equational theory then is a set Σ of identities for which $Id\ Mod\Sigma = \Sigma$, meaning that any identity satisfied by all the algebras which satisfy all the identities in Σ must itself be an identity in Σ. This is equivalent to closure of the set Σ under five basic *rules of deduction* for equational logic:

1. (Reflexive rule): For any term p, we can deduce the identity $p \approx p$.
2. (Symmetry rule): From any identity $p \approx q$ we can deduce $q \approx p$.
3. (Transitive rule): From identities $p \approx q$ and $q \approx r$, we can deduce $p \approx r$.
4. (Compatibility rule): If f_i is an n_i-ary operation symbol of type τ, and we have identities $s_1 \approx t_1, s_2 \approx t_2, \ldots, s_{n_i} \approx t_{n_i}$, then we can deduce $f_i(s_1, s_2, \ldots, s_{n_i}) \approx f_i(t_1, t_2, \ldots, t_{n_i})$.
5. (Substitution rule): Let x_j be a variable which occurs in an identity $p \approx q$, and let t be any term. Let $\overline{p}$ and $\overline{q}$ be the terms formed from p and q respectively by replacing every occurrence of variable x_j in them by t. Then we can deduce $\overline{p} \approx \overline{q}$ from $p \approx q$.

An identity $s \approx t$ is said to be a *consequence* of a set Σ of identities if $s \approx t$ is either in Σ itself or can be deduced from Σ by some finite sequence of steps based on the five rules of deduction. A set Σ is *closed* if any consequence of identities in Σ is in Σ. It follows from properties of the Galois correspondence that for any variety V, the set IdV is closed.
Finally, let us define a basis of a variety. Let V be a variety of some type τ. A set Σ of identities of type τ is called an *equational basis* or simply a *basis* for V if $Id\ Mod\Sigma = IdV$; this simply means that every identity holding in V can be deduced from the identities in Σ using the five rules of deduction. We can always use IdV itself as a basis for V, since certainly every identity in IdV can be deduced from IdV. But this is an infinite set of identities, and usually a smaller, perhaps finite, basis can be found.

2. **Complexity and k-Normality.** We can use the complexity of terms to measure the complexity of identities and in turn the complexity of algebras and varieties. The most commonly used measure of complexity of terms is the *depth* of a term. For any term t, the depth of t is the length of the longest path from the root to a leaf in the tree diagram for t. The term t shown in Figure 1, for example, has depth 3. Formally, for each term t of type τ, we denote by $d(t)$ the depth of t, defined inductively by
(i) $d(t) = 0$, if t is a variable x_j for some $j \geq 1$;
(ii) $d(t) = 1 + max\{d(t_j) : 1 \leq j \leq n_i\}$, if t is a composite term $t = f_i(t_1, \ldots, t_{n_i})$.
The depth function is an example of a valuation function on the set of all terms of type τ (see [2]).
Let $k \geq 0$ be any natural number. An identity $s \approx t$ of type τ is called *k-normal* (with respect to the depth valuation) if either s and t are equal, or $d(t)$, $d(s) \geq k$. We denote by $N_k(\tau)$ the set of all k-normal identities of type τ. It was proved in [2] that the set $N_k(\tau)$ is closed under the usual five rules of deduction for identities and so is an equational theory.
For a variety V of type τ, we let $Id\,V$ denote the set of all identities of V. The set $Id_k\,V = N_k(\tau) \cap Id\,V$ of all k-normal identities satisfied by V is also an equational theory. Then $Mod\,Id_k\,V$ is a variety, called the *k-normalization of* V. In the special case that $N_k(V) = V$, we say that V is a *k-normal variety*; this occurs when every identity of V is a k-normal identity. Otherwise, V is a proper subvariety of $N_k(V)$, and $N_k(V)$ is the least k-normal variety to contain V. This is a generalization of the well-known property of *normality* of identities and varieties (see [3], [4]), which coincides with our k-normality for $k = 1$.
The variety $N_k(V)$ is defined equationally, by means of the k-normal identities of V. An algebraic characterization of the algebras in $N_k(V)$ was given by Denecke and Wismath in [1], using the concept of a k-choice algebra. They showed that any algebra in $N_k(V)$ is a homomorphic image of a k-choice algebra constructed from an algebra in V.
In this paper we return to the equational approach, to study identities that are k-normal consequences of associativity. A semigroup is an algebra of type (2), that is having one binary operation symbol, which satisfies the associative law. We use Sem for the variety of all semigroups, and note that $Sem = Mod\Sigma$ for the one-element set Σ consisting of the associative identity. Since both sides of the associative identity have depth 2, and the same is true for all consequences of associativity, the variety Sem is both 1-normal and 2-normal, and

$N_1(Sem) = N_2(Sem) = Sem$, with Σ as a basis. For $k \geq 3$ however, the k-normalization varieties $N_k(Sem)$ no longer satisfy associativity, but only its k-normal consequences. The goal of this paper is to produce a finite set of identities which is an equational basis of $N_k(Sem)$, for $k \geq 3$.

3. **The Basis Theorem.** In this section we introduce some notation and terminology needed to state our main theorem. First, the variety Sem is a type (2) variety, having one binary operation symbol f. We shall follow the usual convention regarding terms of this type, by which we often omit the symbol f from terms and tree diagrams, and denote the operation by juxtaposition instead. For instance, we can write the associative identity $f(f(x_1, x_2), x_3) \approx f(x_1, f(x_2, x_3))$ simply as $(x_1x_2)x_3 \approx x_1(x_2x_3)$. In an associative setting we could also omit the brackets in terms, but in non-associative varieties such as $N_k(Sem)$ for $k \geq 3$ the brackets are necessary to indicate the grouping of variables in a term.

In general, for any binary term t, we call the sequence $x_{i_1}x_{i_2}\ldots x_{i_p}$ of the variables occurring in t, in the order in which they occur in t from left to right, the *underlying word* of t. This word can be obtained by writing the term t with the operation represented by juxtaposition, and with all brackets omitted from t. A key fact about semigroup identities is that an identity $s \approx t$ is a consequence of associativity if and only if the underlying words obtained from s and t are equal.

To produce our basis for $N_k(Sem)$ we shall make use of a certain kind of terms of type (2).

Definition 3.1. Let $k \geq 1$. We shall refer to any term t whose underlying word is $x_1x_2\cdots x_{k+1}$ as a *skeleton term* of depth k. Such terms have exactly k occurrences of the operation symbol f and exactly one occurrence of each of the variables $x_1, \ldots, x_{k+1}$, in that order from left to right, and no other variables. We shall denote by Γ_k the set of all the skeleton terms of depth k.

For $k = 1$, there is only one skeleton term, the term x_1x_2; for $k = 2$ there are two skeleton terms, $x_1(x_2x_3)$ and $(x_1x_2)x_3$. We shall make frequent use of the four skeleton terms for $k = 3$, labelled as t_1, t_2, t_3 and t_4 and shown in Figure 2. It is easy to show by induction on k that for $k \geq 1$, there are 2^{k-1} skeleton terms.

Our basis for $N_k(Sem)$ will essentially consist of identities which say that any two skeleton terms of depth k should be equivalent to each other. We can make the basis smaller by picking one skeleton term and ensuring that any other term is equivalent to it. The one skeleton we choose to focus on will be called the *ladder* term of depth k. It is a special case of a general shape called a ladder shape.

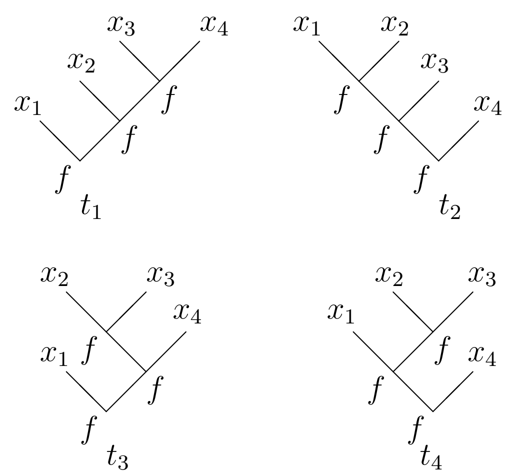


Figure 2: Skeleton Terms for $k = 3$

Definition 3.2. A ladder term is any term which has the form $f(x_{i_1}, f(x_{i_2}, f(x_{i_3}, \ldots, f(x_{i_j}, x_{i_{j+1}})) \cdots)$, for some variables $x_{i_1}, \ldots, x_{i_{j+1}}$. We denote by l_k the particular ladder term

$$l_k = l_k(x_1, \ldots, x_{k+1}) \;=\; f(x_1, f(x_2, f(x_3, \ldots, f(x_k, x_{k+1})) \ldots),$$

shown in Figure 3 below.

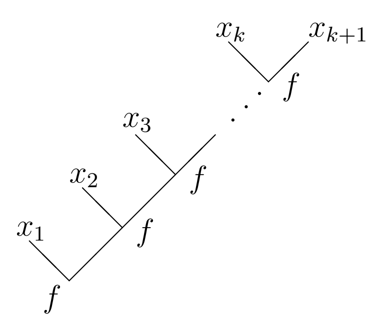


Figure 3: The ladder term l_k

Let t be a term with underlying word $x_{i_1} \cdots x_{i_p}$. We call the term $l(t) = f(x_{i_1}, f(x_{i_2}, f(x_{i_3}, \ldots, f(x_{i_j}, x_{i_{j+1}})) \cdots)$ the ladder of t. Clearly $t \approx l(t)$ is a consequence of associativity.

Theorem 3.3 (The Basis Theorem)**.** *For $k \geq 3$, the set $\Sigma_k = \{l_k \approx w \mid w \in \Gamma_k\}$ forms a finite basis for the identities of $N_k(Sem)$.*

It is clear that by using the deduction rules of transitivity and symmetry we can deduce from Σ_k any identity $v \approx w$ where v and w are skeleton terms of depth k. We shall henceforth assume that all such identities are in our basis Σ_k.

4. **Preliminary Lemmas and the Base Case.** We shall prove the Basis Theorem by induction on k, for $k \geq 3$. In this section we verify that the basis works for the base case of the induction, the case $k = 3$; but first we prove some preliminary results. The first lemma shows that if we have a ladder term juxtaposed with another term, we can "merge" the second term into the ladder shape.

Lemma 4.1 (The Merging Lemma)**.** *Let $n \geq 2$. Let s_1 be a ladder term*

$$s_1 \;=\; y_1(\cdots(y_n(y_{n+1}(\cdots(y_j y_{j+1})\cdots)$$

for some variables $y_1, \ldots, y_{j+1}$ and some depth $j \geq n$. Let s_2 be any term, and let $s = s_1 s_2$. Then we can use the identities in Σ_{n+1} to deduce $s \approx t$, where $t = y_1(y_2(\cdots y_j(y_{j+1}, s_2)\cdots)$. (See Figure 4 for illustration.)

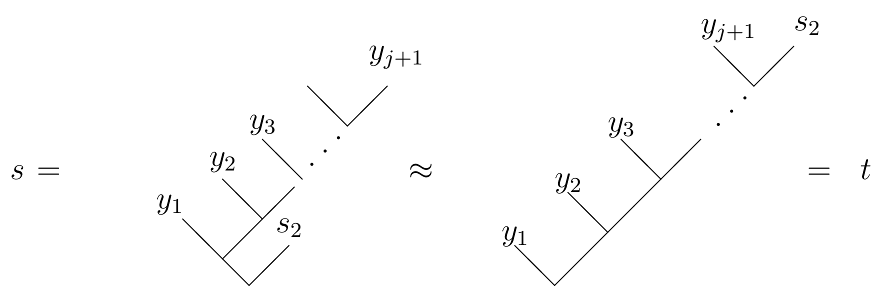


Figure 4: Merging s_2 into the ladder s_1

Proof: We consider first the skeleton term

$$p \;=\; (x_1(x_2(\cdots(x_n x_{n+1})\cdots)x_{n+2},$$

of depth $n+1$. We can regard our term s as an instance of this skeleton, if we replace x_i by y_i for $1 \leq i \leq n$ and x_{n+2} by the term s_2 and x_{n+1} by the term $b_1 = y_{n+1}(y_{n+2}(\cdots(y_j y_{j+1}))\cdots)$. In terms of the rules of deduction, we consider the identity from Σ_{n+1} which equates the skeleton term p with the ladder skeleton term l_{n+1}. Using the substitution rule to replace variables by terms, as just described, we see that we can deduce $s \approx s'$, where s' is the term $y_1(y_2(\cdots(y_n(b_1 s_2)\cdots)$. If $j = n+1$, we have $s' = t$, and our proof is complete. Otherwise, we continue this process as follows. We can write $s' = y_1 w$ for some term w of depth at least $n+1$, and we can view this new term w as an instance of a skeleton term of depth $n+1$. Then we can use the substitution rule again, on the basis identity which equates that skeleton with the ladder skeleton term, to deduce $s' \approx s'' = y_1(y_2(y_3(\cdots(y_n(y_{n+1}(b_2 s_2)\cdots)$. Continuing in this way $j-(n+1)$ times, moving up the tree diagram for the term one place each time, we reach the term t. Then by transitivity we can deduce $s \approx t$. □

Corollary 4.2 (Merging Ladders Corollary). *Let $n \geq 2$. Let $s = s_1 s_2$ where s_1 is a ladder with depth $\geq n$ and s_2 is any ladder term. Then $s \approx l(s)$ can be deduced from Σ_{n+1}.*

Now we can prove the base case in the inductive proof of our main theorem.

Lemma 4.3 (Basis for $N_3(Sem)$). *The set Σ_3 is a basis for $N_3(Sem)$.*

Proof: To show that Σ_3 is a basis for $N_3(Sem)$, we need to show that any 3-normal identity which holds in Sem can be deduced from Σ_3. Let $s \approx t$ be any such identity. Then since $s \approx t$ holds in Sem we must have $l(s) = l(t)$. So it will suffice to prove that for any term s of depth ≥ 3, we can deduce the identity $s \approx l(s)$ from Σ_3. We prove this claim by induction on the depth n of the term s. Let us recall that Σ_3 consists of all the identities $t_i \approx t_j$ for $1 \leq i, j \leq 4$, using the four depth 3 skeletons from Figure 2.
For the base case of our induction on n, we need to show that for any of the 21 terms s of depth exactly 3, we can deduce $s \approx l(s)$. Since most of the deductions are quite similar in nature, we illustrate here with one example, and leave the details of the remaining 20 cases for the Appendix. Let $s = ((x_1x_2)(x_3x_4))x_5$ be the term shown in Figure 5 below. We apply the substitution rule of deduction to the basis identity $t_2 \approx t_1$, to deduce $s \approx x_1(x_2(x_3x_4)x_5)$. Then we use the compatibility rule on the two identities $x_1 \approx x_1$ and $x_2((x_3x_4)x_5) \approx x_2(x_3(x_4x_5))$, the latter of which is an instance of $t_3 \approx t_1$, to deduce $x_1(x_2(x_3x_4)x_5)$ $\approx l(s)$. Now we have $s \approx l(s)$ by transitivity. The two steps in this deduction are shown in Figure 5, where we indicate in brackets each time which basis identity we are using.
Now for the inductive step, we assume for $s \approx l(s)$ can be deduced from Σ_3 for any s of depth $n \geq 3$, and we consider terms of depth $n+1$. If s is a term of depth $n+1$, we can write $s = s_1 s_2$ for some terms s_1 and s_2, at least one of which must have depth at least n. We consider two cases, depending on whether s_1 or s_2 has depth at least n.

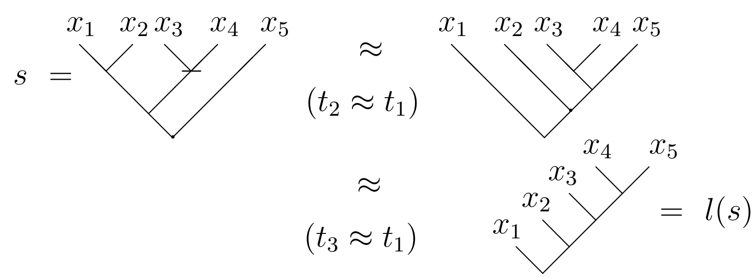


Figure 5: Deduction of $s \approx l(s)$

CASE 1: $d(s_2) = n \geq 3$:
Then by assumption, $s_2 \approx l(s_2)$ can be deduced from Σ_3. There are four subcases to consider, depending upon the depth of the other term s_1.
a) If $d(s_1) = 0$, then s_1 is a variable, and in this case $s = s_1 s_2 \approx s_1 l(s_2) = l(s)$.
b) If $d(s_1) = 1$, then $s_1 = y_1 y_2$ for some variables y_1 and y_2. Since by assumption $s_2 \approx l(s_2)$, we have $s \approx (y_1 y_2) l(s_2)$ by the compatibility rule of deduction. We can write $l(s_2) = w_1(w_2(\cdots(w_j w_{j+1})\cdots)$ for some variables $w_1, \ldots w_{j+1}$ with $j \geq n$. Let $b = w_3(w_4(\cdots(w_j w_{j+1})\cdots)$. Thus we have $s \approx s'$ where $s' = s_1(w_1(w_2 b))$. Now we perform a series of three substitutions on identities from Σ_3, to transform s' into $l(s)$, as illustrated in Figure 6 below. First consider the identity $t_1 \approx t_4$. By the substitution of s_1 for x_1, w_1 for x_2, w_2 for x_3 and b for x_4, we get $s' \approx s''$ where s'' is the term $(y_1 y_2)((w_1 w_2) b)$. Next we use the identity $t_2 \approx t_1$, and the substitution of y_1 for x_1, y_2 for x_2, $w_1 w_2$ for x_3 and b for x_4, to get $s'' \approx s'''$, where $s''' = y_1(y_2((w_1 w_2) b))$. Finally, we use $t_3 \approx t_1$, and substitute y_2 for x_1, w_1 for x_2, w_2 for x_3 and b for x_4, to deduce $y_2((w_1 w_2) b)) \approx y_2(w_1(w_2 b))$. Using the compatibility rule to left-multiply both sides of this last identity by y_1, we get $s''' \approx y_1(y_2(w_1(w_2 b))) = l(s)$. Overall, by transitivity we have deduced $s \approx l(s)$.
c) If $d(s_1) = 2$, then there are 3 subcases to consider for the shape of s_1.
i) If $s_1 = (y_1 y_2) y_3$ for some variables y_1, y_2 and y_3, then we consider the identity $t_2 \approx t_1$ from Σ_3. Using $s_2 \approx l(s_2)$ by induction, and the substitution of y_i for x_i for $1 \leq i \leq 3$ and $l(s_2)$ for x_4, we get $s \approx y_1(y_2(y_3 l(s_2))) = l(s)$.

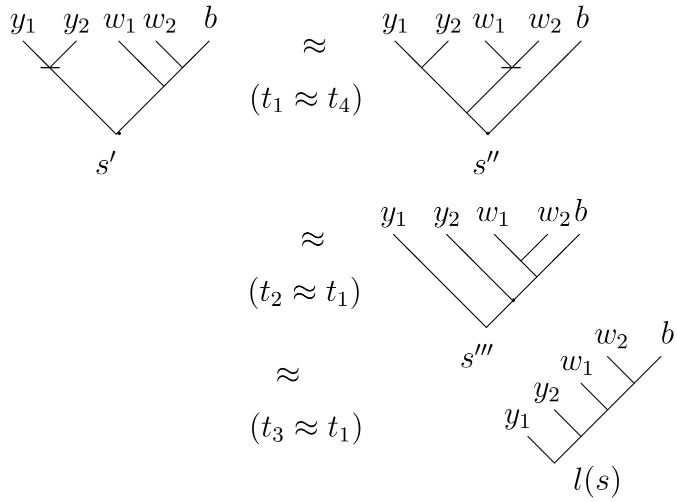


Figure 6: Deduction of $s \approx l(s)$ in Case 1b)

ii) If $s_1 = (y_1y_2)(y_3y_4)$ for some variables $y_1, \ldots, y_4$, then we consider first the identity $t_2 \approx t_1$. Making the substitution of y_1 for x_1, y_2 for x_2, y_3y_4 for x_3 and s_2 for x_4, we deduce $s \approx s' = y_1(y_2((y_3y_4)s_2))$. Next we substitute into $t_3 \approx t_1$, using y_2 for x_1, y_3 for x_2, y_4 for x_3 and s_2 for x_4. This yields the identity $y_2((y_3y_4)s_2) \approx y_2(y_3(y_4(s_2)))$. Using the compatibility rule to introduce y_1 on the left side of each term in this last identity, we get $y_1(y_2((y_3y_4)s_2)) \approx y_1(y_2(y_3(y_4(s_2))))$. The left hand side of this identity is s'; on the right hand side we have a ladder shape with s_2 in the last position on the ladder. Since by induction we have $s_2 \approx l(s_2)$, we get $s \approx y_1(y_2(y_3(y_4(l(s_2))))) = l(s)$.
iii) If $s_1 = y_1(y_2y_3)$ for some variables y_1, y_2 and y_3, we use the identity $t_4 \approx t_1$. Substitution of y_i for x_i, for $1 \leq i \leq 3$, and s_2 for x_4, gives $s \approx y_1(y_2(y_3(s_2)))$. Using the compatibility rule on the assumption that $s_2 \approx l(s_2)$ then gives $s \approx l(s)$.
d) If $d(s_1) = n \geq 3$, then by the induction assumption we can deduce both $s_1 \approx l(s_1)$ and $s_2 \approx l(s_2)$. Then we get $s = s_1s_2 \approx l(s_1)l(s_2)$. By the Merging Ladders Corollary above, with $n = 2$, we can deduce $l(s_1)l(s_2) \approx l(s)$ from Σ_3, giving us $s \approx l(s)$.
CASE 2: $d(s_1) = n \geq 3$:
Then by assumption, $s_1 \approx l(s_1)$ can be deduced from Σ_3. If s_2 is also a ladder, then we can deduce $s = s_1s_2 \approx l(s)$ from Σ_3, by the Merging Ladders Corollary 4.2. This happens if s_2 is a variable or a depth 1 term of the form y_1y_2 for some variables y_1 and y_2. If s_2 has depth 3 or more, then by induction we can ladder it to get $s_2 \approx l(s_2)$, and then use Corollary 4.2 again to get $s = s_1s_2 \approx l(s_1)s_2 \approx l(s_1)l(s_2) \approx l(s_1s_2)$ $= l(s)$. This leaves only the subcase that s_2 is a term of depth 2.

There are three possible shapes for a term of depth 2. If $s_2 = y_1(y_2y_3)$ for variables y_1, y_2, y_3, then s_2 is a ladder, and we can use the Merging Ladders Corollary 4.2 to deduce $s = s_1s_2 \approx l(s_1)l(s_2) \approx l(s_1s_2) = l(s)$. If $s_2 = (y_1y_2)y_3$ for variables y_1, y_2, y_3, then we use the identity $t_3 \approx t_1$. By the substitution of s_1 for x_1, y_1 for x_2, y_2 for x_3 and y_3 for x_4, we deduce $s = s_1((y_1y_2)y_3) \approx s_1(y_1(y_2y_3))$. Then we use the induction hypothesis that $s_1 \approx l(s_1)$ to deduce $s_1(y_1(y_2y_3)) \approx l(s_1)(y_1(y_2y_3)) = l(s)$. By transitivity then we have $s \approx l(s)$. Finally, if $s_2 = (y_1y_2)(y_3y_4)$ for some variables y_j, we substitute s_1 for x_1, y_1 for x_2, y_2 for x_3 and y_3y_4 for x_4 into the identity $t_3 \approx t_1$. This yields the identity $s \approx s_1(y_1(y_2(y_3y_4)))$. Then using the assumption to make $s_1 \approx l(s_1)$ and the Merging Ladders Corollary 4.2 again, we get $s \approx l(s)$. □

5. **Proof of the Basis Theorem.** To finish the proof of our Basis Theorem, we need one additional property about the sets Σ_k of identities. These sets have a sort of "nested" property, in the following sense. If s is a skeleton term of depth k, for some $k \geq 2$, then (possibly after some relabelling of variables) the two terms xs and sx are skeleton terms of depth $k+1$, for any variable x. Thus if t_i and t_j are skeleton terms of depth k, with $t_i \approx t_j$ in Σ_k, we have the identities $x\ t_i \approx x\ t_j$ and $t_i\ x \approx t_j\ x$ in Σ_{k+1}.

Now we are ready to prove our main Basis Theorem, Theorem 3.3. We want to prove that for any $k \geq 3$, any identity $s \approx t$ which holds in $N_k(Sem)$, that is, holds in Sem and has both $d(s)$ and $d(t)$ at least k, can be deduced from the set Σ_k. Since an identity $s \approx t$ holds in Sem if and only if $l(s) = l(t)$, it is sufficient to prove that for any $k \geq 3$ and any term s of depth $\geq k$, we can deduce $s \approx l(s)$ from Σ_k. We proceed by (strong) induction on k. The base case, $k = 3$, holds by Lemma 4.3. Inductively, we assume that for any term t of depth j we can deduce $t \approx l(t)$ from Σ_j, for $3 \leq j \leq k$. Using this we show how we can deduce $s \approx l(s)$ for any term s of depth $\geq k+1$ using Σ_{k+1}.

Let s be any term of depth $\geq k+1$. This means that we can write $s = s_1s_2$ for some terms s_1 and s_2, at least one of which must have depth at least k.

CASE 1: $d(s_1) \geq k$:

By the inductive assumption, we can deduce $s_1 \approx l(s_1)$ from Σ_k. Then we can use the nested property described above to deduce $s = s_1s_2 \approx l(s_1)s_2$ from Σ_{k+1}: each time an identity $t_i \approx t_j$ was used in the deduction of $s_1 \approx l(s_1)$ from Σ_3, we now use use $t_i\ x \approx t_j\ x$ from Σ_{k+1}. Let us write $l(s_1) = x_1(x_2(\cdots x_p)\cdots)$. Then by Lemma 4.1, with $n = k$, we can use Σ_{k+1} to deduce $s \approx x_1(\cdots(x_{n-1}(x_ps_2)\cdots)$.

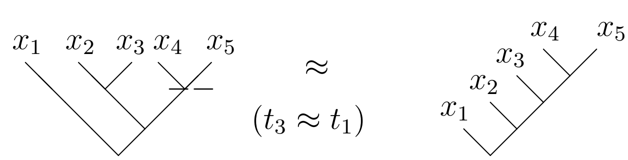
This last term has a ladder shape if s_2 has a ladder shape, in particular if s_2 is a variable or has depth 1. If $d(s_2) = 2$, then we can deduce $x_p s_2 \approx l(x_p s_2)$ using only identities of the form $t_i \approx t_j$ from Σ_3. We can then deduce $w = x_1(x_2(\cdots(x_p s_2)\cdots)) \approx l(w)$, using the nested version $x_1(x_2(\cdots(x_{p-1}t_i\cdots))) \approx x_1(x_2(\cdots(x_{p-1}t_j\cdots)))$ in Σ_{k+1}.
Similarly, if $d(s_2) \geq 3$ we can deduce $s_2 \approx l(s_2)$ using some identities $t_i \approx t_j$ from Σ_3, and so deduce $s \approx l(s)$ by the corresponding nested identities $x_1(\cdots(x_n t_i)\cdots) \approx x_1(\cdots(x_n t_j)\cdots)$ in Σ_{k+1}.

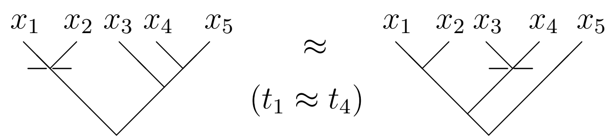
CASE 2: $d(s_2) \geq k$:
By the inductive assumption, we can deduce $s_2 \approx l(s_2)$ from Σ_k, and again by using the corresponding nested versions of any identities used in this deduction, we are able to deduce $s \approx s_1 l(s_2)$ from Σ_{k+1}. Then if s_1 is a variable only, we have a ladder and $s \approx l(s)$. So we suppose that $d(s_1) \geq 1$. We can write $l(s_2)$ as $y_1(y_2(\cdots(y_{n-1}y_n)\cdots)$ for some variables $y_1, \ldots, y_n$. Then we can use one of the skeleton identities from Σ_{k+1} to deduce $s \approx s_1' s_2' = (\cdots(s_1 y_1)y_2)\cdots)y_{k-1})(y_k(\cdots(y_{n-1}y_n)\cdots)$. Since $d(s_1') \geq k$, we can now apply Case 1, to conclude that we can deduce $s \approx l(s)$ as required. □

APPENDIX: THE 21 BASE CASES FOR LEMMA 4.3
Here we show the details of the deduction of $s \approx l(s)$ from Σ_3, for each of the 21 possible terms s of depth 3. In each case we use substitution and compatibility on one or more of the basis identities from Σ_3. We indicate by the symbol $--$ any place where we are substituting a larger term for a single variable in a skeleton term; and we use $\circ$ to indicate a portion of a tree which we intend to regard as a skeleton term of depth 3 in order to then use compatibility. The terms s are grouped according to whether they use 4, 5, 6, 7 or 8 variables.

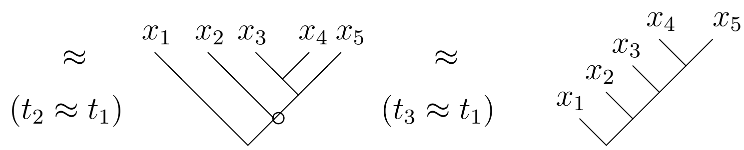
Terms with Four Variables: These are the four skeleton terms, and each $t_i \approx t_j$ is already in Σ_3.

Terms with Five variables:

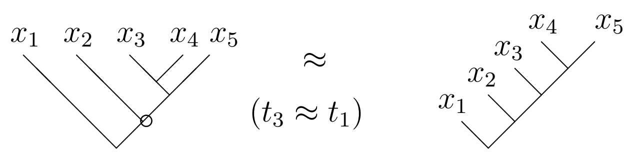




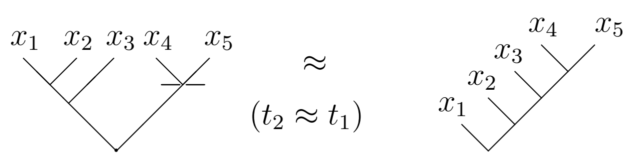
x_1 x_2 x_3 x_4 x_5
≈
$(t_1 \approx t_4)$
x_1 x_2 x_3 x_4 x_5



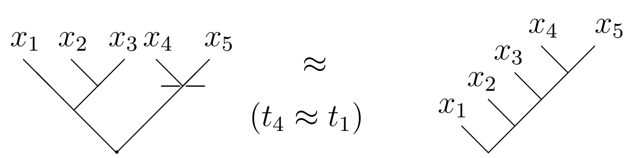
≈
$(t_2 \approx t_1)$
x_1 x_2 x_3 x_4 x_5
≈
$(t_3 \approx t_1)$
x_1 x_2 x_3 x_4 x_5



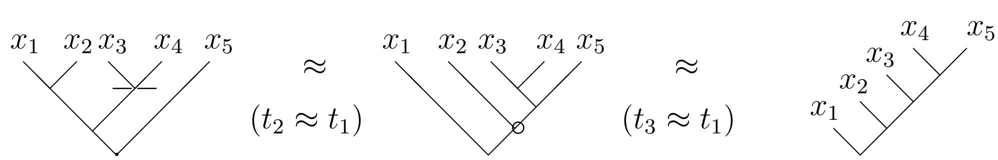
x_1 x_2 x_3 x_4 x_5
≈
$(t_3 \approx t_1)$
x_1 x_2 x_3 x_4 x_5



x_1 x_2 x_3 x_4 x_5
≈
$(t_2 \approx t_1)$
x_1 x_2 x_3 x_4 x_5

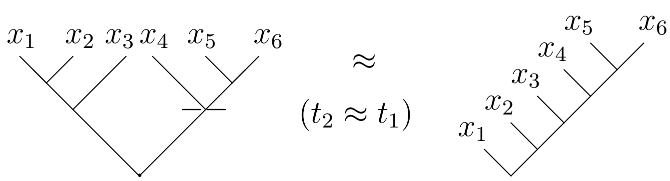


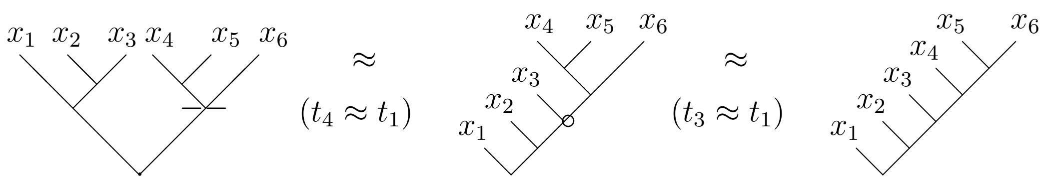
x_1 x_2 x_3 x_4 x_5
≈
$(t_4 \approx t_1)$
x_1 x_2 x_3 x_4 x_5

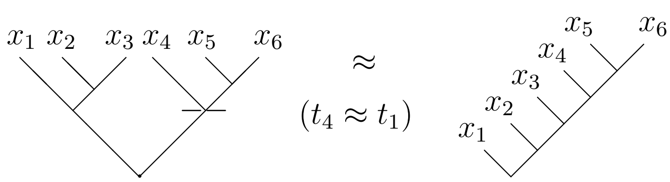


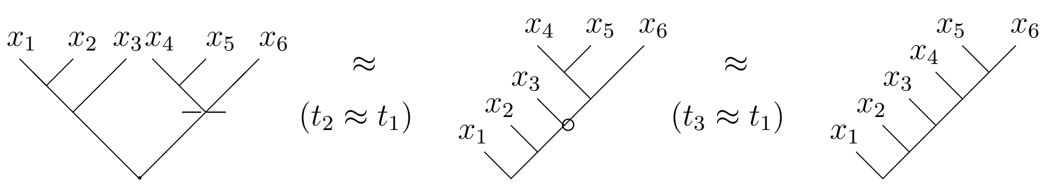
x_1 x_2 x_3 x_4 x_5
≈
$(t_2 \approx t_1)$
x_1 x_2 x_3 x_4 x_5
≈
$(t_3 \approx t_1)$
x_1 x_2 x_3 x_4 x_5

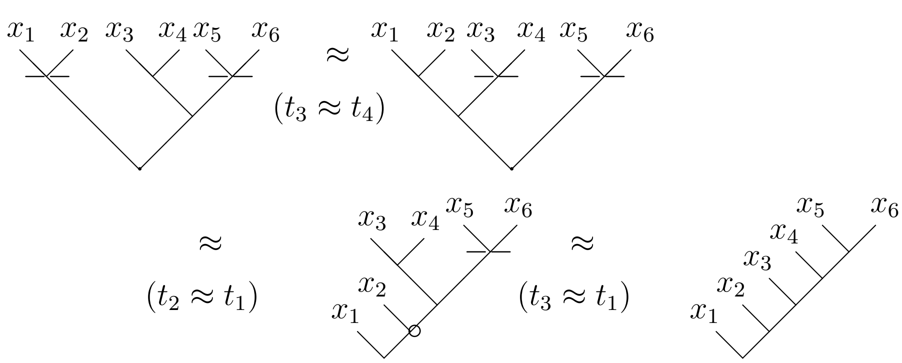
Terms with Six Variables :

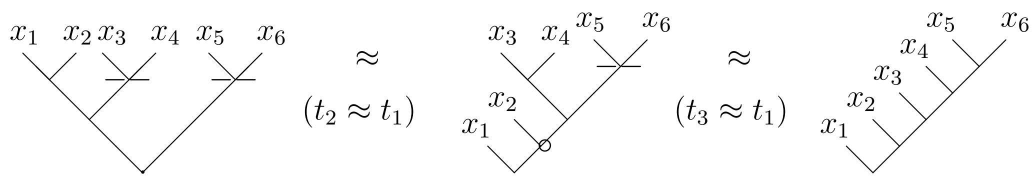




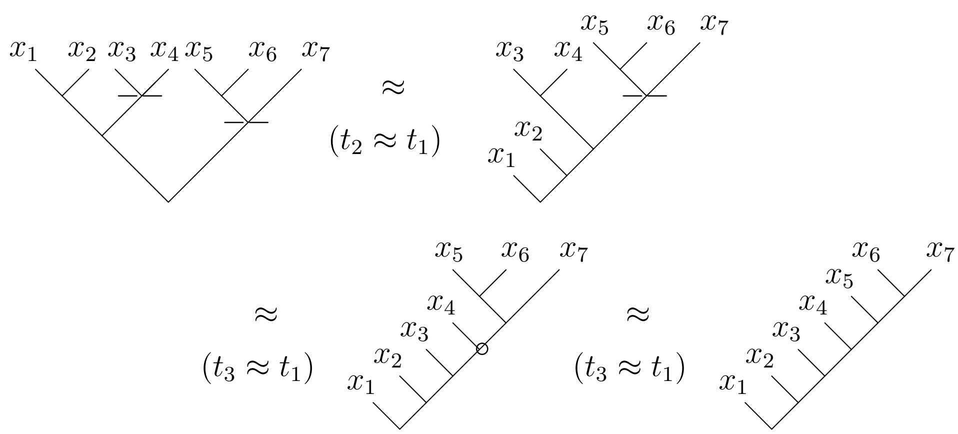


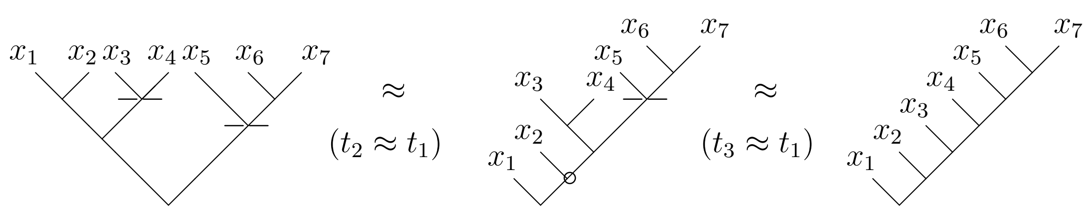


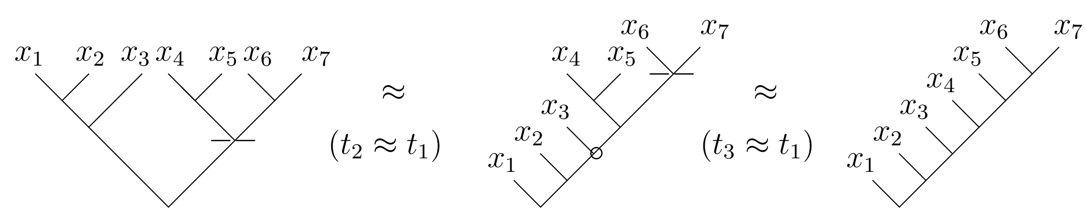


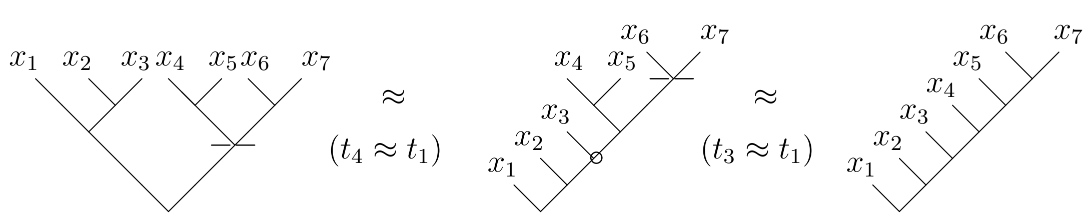


Terms with Seven Variables :

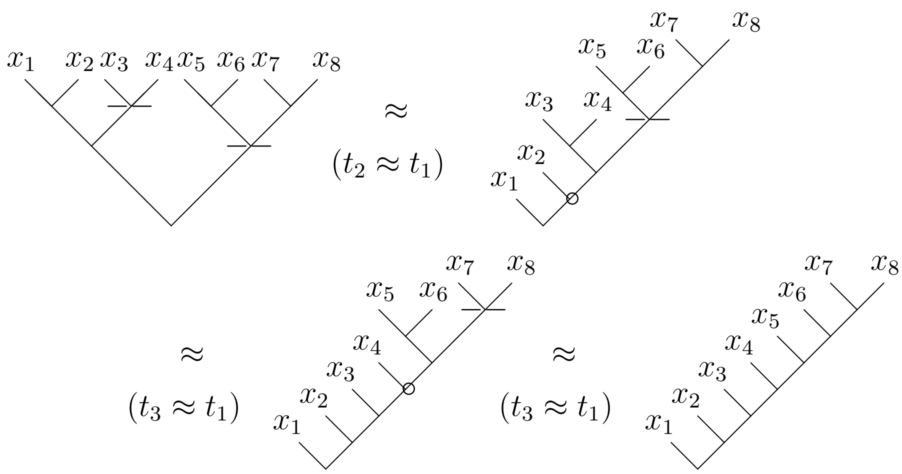








Terms with Eight Variables :



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