Atlantic Electronic
Journal of Mathematics
Volume **1**, Number **1**, Summer **2006**

http://aejm.ca
http://rema.ca
pp. **5–30**

A SURVEY OF GRACEFUL TREES

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Abstract. A tree of order n is said to be graceful if the vertices can be assigned the labels $\{0, \ldots, n-1\}$ such that the absolute value of the differences in vertex labels between adjacent vertices generate the set $\{1, \ldots, n-1\}$. The Graceful Tree Conjecture is the unproven claim that all trees are graceful. We present major results known on graceful trees from those dating from the problem's origin to recent developments. Constructions and classes of graceful trees are given as well as an analysis of various lines of attack on the Conjecture. We also examine other types of graph labellings as they relate to the Conjecture.

1. **Introduction.** A graph $G = (V, E)$ is said to be *labelled by* ϕ if each vertex $v \in V$ is assigned a non-negative integer value $\phi(v)$, and each edge $e = uv \in E$ is assigned the value $|\phi(u) - \phi(v)|$. The labelling ϕ is *graceful* if $\phi : V \to \{0, 1, 2, ..., |E|\}$ is an injection and if all edges of G have distinct labels from $\{1, 2, ..., |E|\}$. A graph is *graceful* if it admits a graceful labelling, also known as a *graceful numbering* or *valuation*. *Bipartite labellings* have the further property that there exists $x \in \{0, 1, \ldots, n\}$, such that for an arbitrary edge $v_i v_j$ of the graph G either $a_i \leq x$, $a_j > x$ or $a_i > x$, $a_j \leq x$ holds (where a_k is the label assigned to vertex x_k). Refer to [23] for additional notation and terminology not defined here.

The origins of graceful labellings lie in the problem of packing isomorphic copies of a given tree into a complete graph. Graceful labellings can also be used to study a classical combinatorial problem. Golomb [30] poses the question of how to notch a metal bar k units in length at a minimum number of integer points in such a way that the distances between any two notches, or between a notch and an endpoint, are distinct and generate the set $\{1, ..., k\}$.

No *general* method is currently known to allow one to take a tree known to be graceful and to extend a path from it in an arbitrary position, or to identify an arbitrary vertex with another general tree known to be graceful, in order to produce a larger, graceful tree. The fact that not all trees admit labellings in which a leaf is labelled 0 has been a major stumbling block to proving that all trees are graceful. If a leaf of any tree could be labelled 0 under a graceful labelling, one could inductively generate graceful labellings for arbitrary trees. We survey various

2000 *Mathematics Subject Classification.* Primary: 05C78; Secondary: 05C75.
Key words and phrases. Graceful, Graceful Tree Conjecture, graph labelling.

operations that generate large classes of graceful trees from smaller graceful trees and briefly inventory classes of graceful trees known. We discuss the generalization of the graceful labelling problem to graphs including cycles, wheels and cones.

We also discuss variations on graceful labellings. The motivation of much of the work in this area has been the hope that solving conjectures related to other vertex labellings could produce new techniques adaptable to attacks on the conjecture that all trees are graceful. Asymptotics and bounds relating to graceful graphs are mentioned, and the possibility of a solution to the conjecture from this perspective is discussed.

2. **Graph Decompositions and the Origin of the Graceful Tree Conjecture.** A *decomposition* of a graph G is a collection $\{H_i\}$ of nonempty subgraphs such that $H_i = \langle E_i \rangle$ for some (nonempty) subset E_i of $E(G)$, where $\{E_i\}$ is a partition of $E(G)$. If $\{H_i\}$ is a decomposition of a graph G such that, for each i, $H_i = H$ for some graph H, then G is said to be H*-decomposable.* If G is an H-decomposable graph, then we write $H|G$ and say that H *decomposes* G.

A *cyclic decomposition* is a decomposition of a graph G into k copies of a subgraph H that can be obtained in the following manner:

1. draw G appropriately
2. select a subgraph H_1 of G that is isomorphic to H
3. rotate the vertices and edges of H_1 through an appropriate angle $k-1$ times to produce k copies of H in the decomposition.

If $H|G$ then the size of H necessarily divides the size of G, and H is necessarily a subgraph of G. However the fact that the size of H divides the size of G is not a sufficient condition for $H|G$. For example, $4|12$ but $H \nmid G$ where $G = K_{2,2,2}$ and $H = K_{1,4}$. It is easy to see that every nonempty graph is K_2-decomposable.

There is a long history of studying graph decompositions from a design theory standpoint. For example, a K_3-decomposable complete graph K_n is also known as a *Steiner triple system* (where the vertices of K_n are points in the design, each copy of K_3 is a block of the design and every pair of vertices appears in exactly one block). There is a theorem attributed to Kirkman that characterizes when such a decomposition is attainable:

Theorem 1. [23] *K_n is K_3-decomposable if and only if n is odd and $3|\binom{n}{2}$.*

Whenever K_n is K_k decomposable for natural numbers $k \geq 3$ we have an example of a balanced incomplete block design. Graph decompositions may be viewed as generalized block designs.

R. M. Wilson [78] proved that for every graph H without isolated vertices, there exist infinitely many positive integers n such that K_n is H-decomposable. The following is Wilson's theorem as it appears in [23]:

Theorem 2. *For every graph H without isolated vertices and having size m, there exists a positive integer N such that if (i) $n \geq N$, (ii) $m|\binom{n}{2}$ and (iii) $d|(n-1)$, where $d = \gcd\{deg\, v | v \in V(H)\}$, then K_n is H-decomposable.*

An immediate consequence of this theorem is that there exist regular H-decomposable graphs for every graph H without isolated vertices.

In 1963, at the Symposium held in Smolenice, Ringel posed the following problem:

Conjecture 1. [64] *Every $(m+1)$-order tree decomposes K_{2m+1}.*

This problem has since become known as **Ringel's Conjecture** and is to date still unsolved. It is said [64] that Kotzig conjectured a stronger statement than Ringel's.

Conjecture 2. *Every* $(m+1)$*-order tree cyclically decomposes* K_{2m+1}*.*

Another conjecture is also credited to Ringel; this one is a more general statement than his famous conjecture.

Conjecture 3. [34] *Every tree on* $m+1$ *vertices decomposes* K_{rm+1} *for every natural number* $r \geq 2$ *provided that* r *and* $m+1$ *are not both odd.*

In 1967 Rosa published a paper with the intention of providing insight into attacking Ringel's Conjecture [64]. His idea was to use a labelling of the vertices of a graph H of order m to show that it can cyclically decompose K_{2m+1}. He referred to a labelling as a *valuation* of the graph. Using Rosa's notation, consider a vertex labelling O_G on the graph G. Let V_{O_G} denote the set of numbers assigned to the vertices of G and let E_{O_G} be the set of numbers assigned to the edges of G in the edge labelling induced by O_G. Consider the following conditions:

1. $V_{O_G} \subseteq \{0, 1, \ldots, n\}$,
2. $V_{O_G} \subseteq \{0, 1, \ldots, 2n\}$,
3. $E_{O_G} = \{1, 2, \ldots, n\}$,
4. $E_{O_G} = \{x_1, x_2, \ldots, x_n\}$, where $x_i = i$ or $x_i = 2n+1-i$,
5. there exists $x \in \{0, 1, \ldots, n\}$, such that for an arbitrary edge v_iv_j of the graph G either $a_i \leq x$, $a_j > x$ or $a_i > x$, $a_j \leq x$ holds (where a_k is the label assigned to vertex x_k).

From these conditions Rosa defined four types of labellings:

- A ρ-valuation satisfies conditions (2) and (4).
- A σ-valuation satisfies conditions (2) and (3).
- A β-valuation satisfies conditions (1) and (3).
- An α-valuation satisfies conditions (1), (3) and (5).

Rosa's β-valuation is also known as a graceful labelling (Golomb [30] later introduced the term graceful.) It should be noted that in the literature bipartite labellings and α-valuations are synonymous. If a graph has a bipartite labelling, then it is a bipartite graph.

This definition suggests a hierarchy of labellings; from strongest to weakest it is α-, β-, σ-, and ρ-valuation. Each labelling is a special case of one of its successors in the hierarchy. Using this hierarchy, if it could be shown using the stronger properties that every tree has a bipartite labelling, then of course this would imply that every tree is graceful. However, Rosa showed that not every tree has a bipartite labelling and he was able to classify a family of trees that do not admit such a labelling. Rosa showed the following:

Theorem 3. *If* H *is a graceful graph of size* m*, then* K_{2m+1} *is* H*-decomposable. In fact,* K_{2m+1} *can be cyclically decomposed into copies of* H*.*

Proof: (as appears in [23])

Since H is graceful, there is a graceful labelling of H, that is, the vertices of H can be labelled from a subset of $\{0, 1, \ldots, m\}$ so that the induced edge labels are $1, 2, \ldots, m$. Let $V(K_{2m+1}) = \{v_0, v_1, \ldots, v_{2m}\}$ where the vertices of K_{2m+1} are arranged cyclically in a regular $(2m+1)$-gon, denoting the resulting $(2m+1)$-cycle by C. A vertex labelled i $(0 \leq i \leq m)$ in H is placed at v_i in K_{2m+1} and this is done for each vertex of H. Every edge of H is drawn as a straight line segment in K_{2m+1}, denoting the resulting copy of H in K_{2m+1} as H_1. Hence $V(H_1) \subseteq \{v_0, v_1, \ldots, v_m\}$.

Each edge $v_s v_t$ of K_{2m+1} ($0 \leq s, t \leq 2m$) is labelled $d_C(v_s, v_t)$, where $1 \leq d_C(v_s, v_t) \leq m$. Consequently, K_{2m+1} contains exactly $2m+1$ edges labelled i for each i ($1 \leq i \leq m$) and H_1 contains exactly one edge labelled i ($1 \leq i \leq m$). Whenever an edge of H_1 is rotated through an angle (clockwise, say) of $2\pi k/(2m+1)$ radians, where $1 \leq k \leq m$, an edge of the same label is obtained. Denote the subgraph obtained by rotating H_1 through a clockwise angle of $2\pi k/(2m+1)$ radians by H_{k+1}. Then $H_{k+1} = H$ and a cyclic decomposition of K_{2m+1} into $2m+1$ copies of H results.

□

Although the above theorem says that K_{2m+1} has a cyclic decomposition into every graceful graph H of size m, it is not necessary that H be graceful in order for K_{2m+1} to have a cyclic H-decomposition. For example, C_5 is known to be not graceful (see Section 5) but $C_5|K_{11}$. In his paper Rosa characterized when a graph H of order m cyclically decomposes K_{2m+1}.

Theorem 4. *A cyclic decomposition of K_{2m+1} into subgraphs isomorphic to a graph H of order m exists if and only if there exists a ρ-valuation of the graph H.*

Rosa's work turned concentration toward showing that all trees are graceful in order to prove Ringel's Conjecture. This may in fact have created interest in solving the **Graceful Tree Conjecture**.

Conjecture 4. *All trees are graceful.*

In 1988 Lonc published a paper on resolvable decompositions of graphs [54]. An H-decomposition of G is said to be *resolvable* if the set of graphs $\{H_i\}$ can be partitioned into subsets, called *resolution classes*, such that each vertex of G occurs precisely once in each resolution class. Let rH stand for the disjoint union of r copies of H. Hence a resolvable H-decomposition of G is an rH-decomposition of G, where $r = |V(G)|/|V(H)|$.

Resolvable H-decompositions of K_n have been considered in the following cases:

1. H is a complete graph (when $H = K_3$ this is Kirkman's schoolgirl problem [41])
2. H is a complete bipartite graph
3. H is a cycle (this is the Oberwolfach problem [7])
4. H is a path

The necessary conditions for the existence of a resolvable H-decomposition of K_n are:

$$\begin{aligned} n &\equiv 0 \pmod{|V(H)|} \\ \frac{n(n-1)}{2} &\equiv 0 \quad (\text{mod } \frac{n}{|V(H)|} \cdot |E(H)|) \\ n-1 &\equiv 0 \pmod{d} \end{aligned}$$

where d is the greatest common divisor of the degrees of the vertices in H.

If $H = T$ a tree of order k, then the above three conditions are reduced to:

$$n \equiv k^2 \pmod{\operatorname{lcm}\{2(k-1), k\}}.$$

In his paper Lonc gave necessary and asymptotically sufficient conditions for the existence of a resolvable T-decomposition for every graceful tree T of odd order.

Theorem 5. *Let T be a graceful tree of odd order k. There is an integer $g(T)$ such that if $n \geq g(T)$ then a resolvable T-decomposition of K_n exists if and only if*

$$n \equiv k^2 \pmod{2k(k-1)}.$$

The proof of this theorem uses two lemmas and an algorithm. One of the lemmas follows, but first we need a definition. Let K_n^λ be a complete multigraph of order n such that each edge is repeated λ times.

Lemma 1. *If T is a graceful tree of odd order k, then K_k^2 has a T-decomposition.*

Proof: By the definition of a graceful tree, there is a one-to-one labelling $v : V(T) \to \{1, \ldots, k\}$ such that all the values $|v(x) - v(y)|$, $xy \in E(T)$, are distinct. Without loss of generality we can assume that the set of vertices of K_k^2 is $\{1, \ldots, k\}$. Let $E_0 = \{v(x)v(y) \in E(K_k^2) : xy \in E(T)\}$. The set E_0 induces a tree isomorphic to T in K_k^2. For $p = 1, \ldots, k-1$, we define E_p recursively by $\{(i+1)(j+1) \in E(K_k^2) : ij \in E_{p-1}\}$ (the additions are modulo k). It is easy to verify that $E_0, \ldots, E_{k-1}$ form a T-decomposition of K_k^2.

□

Problems arise when trying to prove that a resolvable T-decomposition of K_k exists when T has an even order k. However Lonc has formulated a generalized conjecture:

Conjecture 5. *For every tree $T \neq K_{1,2p-1}$, $p = 2, 3, \ldots$, there is an integer $g(T)$ such that if $n \geq g(T)$ then a resolvable T-decomposition of K_n exists if and only if*

$$n \equiv k^2 \pmod{\operatorname{lcm}\{2(k-1), k\}}.$$

In 1989 Häggkvist [34] presented an introduction to some standard decomposition problems along with some of his serious research. Many of the results given in his paper use graph labellings as a method of attack. One of the main results is the following theorem. First though, a definition. Let $G(m)$ be the graph obtained by replacing each vertex x by a set X of m new independent vertices $x^1, x^2, \ldots, x^m$ and for every pair of vertices x, y in $V(G)$ joining each vertex in X to each vertex in Y if the edge xy is in $E(G)$.

Theorem 6. *Let T be a tree with m edges and with at least $\frac{m+1}{2}$ leaves. Then $T|K_{2m+1}(3)$.*

One of the main focuses of Häggkvist's research was the decomposition of complete bipartite graphs. Along with Graham, Hağgkvist posed the following conjectures:

Conjecture 6. *Every $(m+1)$-order tree decomposes every $2m$-regular graph.*

Conjecture 7. *Every $(m+1)$-order tree decomposes every m-regular bipartite graph.*

In the literature the latter is referred to as **Häggkvist's Conjecture**.

Häggkvist stated (without proof) some special cases of the above two conjectures.

Theorem 7. *Every $(m+1)$-order tree T with at least $\frac{m+1}{2}$ leaves (for instance every tree without vertices of degree 2) decomposes $C_k(m)$ for $k = 3, 4, \ldots$. Thus in particular T decomposes $K_{m,m,m}$ and $K_{2m,2m}$.*

Theorem 8. *Every $(m+1)$-order tree with diameter at most k decomposes every $2m$-regular graph of girth at least k, for $k = 3, 4, \ldots$.*

Theorem 9. *Every $(m+1)$-order tree with diameter at most $2k$ decomposes every bipartite m-regular graph of girth at least $2k$ for $k = 2, 3, \ldots$.*

More recently, Lladó and López [53] have concentrated on Häggkvist's Conjecture and have been able to show some new progress.

3. **Classes of Graceful Trees.** In 1967 Rosa [64] published the conjecture of Ringel that the complete graph K_{2m+1} can be decomposed into trees isomorphic to a given tree with m edges. Rosa showed that if a tree has a graceful labelling, then the result follows. Attempts to prove Ringel's conjecture have therefore focused on obtaining the stronger result that every tree is graceful, a term coined by Golomb [30]. This is the primary motivation for the results that will be covered in this section.

Let T be a tree and v a vertex of T. A *branch vertex* of T is a vertex of degree at least three in T. A *v-endpath* in T is a path P from v to a leaf of T such that each internal vertex of P has degree two in T. A *spider* $S(a_1, ..., a_r)$ is a tree with exactly one branch vertex v and v-endpaths of lengths $1 \leq a_1 \leq ... \leq a_r$, where $r = \deg v$. The *legs* of a spider are the v-endpaths.

The *parity set* (v) of a vertex v in a tree T is the set of vertices which are an even distance in T from v. The *base* of T under a labelling θ is the vertex b where $\theta(b) = 0$. A labelling θ of T is a *parity valuation* if it induces a bijection between the labels of the vertices in (b) and $\{1, 2, \ldots, p\}$ where b is the base of T under θ and p is the cardinality of (b). An *interlaced labelling* of a tree is a graceful labelling that admits a parity valuation. A tree is *interlaced* if it admits an interlaced labelling.

A tree T is said to be *balanced* if and only if there is an integer r such that whenever u and v are adjacent in T, either $f(u) \leq r < f(v)$ or $f(v) \leq r < f(u)$ is satisfied, where $f(x)$ is the label assigned to the vertex x. Interlaced trees are balanced as they have a bipartite labelling where the labels in one partite set are all larger than any label in the other set. Bloom [14] notes that not all trees are balanced; were this true the gracefulness of all trees would follow. In particular, $K_{1,3}$ with an extra vertex joined to each leaf, known as the spider $S(2, 2, 2)$, is not balanced. Interlaced labellings were introduced by Koh, Tan and Rogers in 1978 [44] and are a special case of the bipartite labellings introduced originally by Rosa [64] in 1967.

A *caterpillar* $C = (X \cup Y, E)$ is a tree consisting of a path $P(C)$ with vertex set X and vertices Y not on $P(C)$, each joined to exactly one vertex of $P(C)$. Original results on the gracefulness of caterpillars are due to Rosa [64]. Let the *base* of a tree T be the tree obtained by deleting the leaves of T and any edges incident with these vertices. In 1967, Rosa showed that any tree that is a path or has a path as its base is graceful by constructing a bipartite labelling of such trees. Since a caterpillar is transformed into a path by the deletion of its leaves, the gracefulness of caterpillars follows by Rosa's construction. The bipartite labelling of a path is achieved by alternately labelling its vertices with the largest and smallest labels possible. In order to label a tree whose base is a path, but is not a path itself, Rosa first labels the first vertex of the path and then any adjacent vertices not on the path, before returning to the next vertex on the path. In this case any collection of vertices not on the path are labelled with consecutive integers; the high-low alternation is resumed when these vertices are all labelled and the next vertex on the path is reached. In the literature, Rosa uses the term *snake* to denote a path, although *path* was already the common usage.

Chen, Lu and Yeh [24] use interlaced labellings to provide an alternative proof to the gracefulness of caterpillars. Using induction on the caterpillar's order, Chen et al. establish that all caterpillars are interlaced with one leaf of $P(C)$ labelled 0 and the other labelled $|X| - 1$ or $|X|$, depending on the parity of the order.

Chen, Lu and Yeh [24] provide the following new classes of graceful trees:

1. A *banana tree* is a vertex v joined to one leaf of any numbers of stars. Let $(2K_{1,1}, ..., 2K_{1,n})$ be the tree obtained by adding a vertex to the union of two copies of each of $K_{1,1}$,...,$K_{1,n}$ and joining it to a leaf of each star. The banana tree obtained in this way is interlaced and therefore graceful. The authors conjecture that all banana trees are graceful. More generally, the banana tree $(a_1K_{1,1}, ..., a_{t-1}K_{1,t-1}, a_tK_{1,t}, K_{1,t+1}, ..., a_nK_{1,n})$ denotes the tree obtained by adding a vertex (the *apex*) to the union of a_i copies of the stars $K_{1,i}$, and joining the vertex to a leaf of each star. Banana trees will be seen further in the work of Bhat-Nayak and Deshmukh following.
2. A *firecracker* F is a tree consisting of a path $P(F)$ and a collection of stars, where each vertex on $P(F)$ is joined to the central vertex of exactly one star. All firecrackers are graceful.
3. A *lobster* L is a tree consisting of a path $P(L)$ and vertices not on $P(L)$ at distance at most two from a vertex of $P(L)$. Hence a tree is a lobster if, after deleting all of its leaves, a caterpillar remains. Lobsters can be thought of as a more general firecracker. A small class of interlaced lobsters is constructed by joining *symmetrical* trees (rooted trees in which every level contains vertices of the same degree) successively at the roots to form a path containing these roots.
4. A *spraying pipe* is a path $v_1, ..., v_n$, where each vertex v_i is joined to m_i paths at a leaf of each path, and where all paths have fixed length. A spraying pipe is interlaced if n is even and $m_{2i-1} = m_{2i}$ for every $1 \leq i \leq \frac{n}{2}$.

Bhat-Nayak and Deshmukh [12] have constructed three new families of graceful banana trees using an algorithmic labelling proof. Extending the results of Chen et al. [24], they have shown the following to be graceful:

(i) $(K_{1,1}, ..., K_{1,t-1}, (\alpha + 1)K_{1,t}, K_{1,t+1}, ..., K_{1,n})$, where $0 \leq \alpha < t$;
(ii) $(2K_{1,1}, ..., 2K_{1,t-1}, (\alpha + 2)K_{1,t}, 2K_{1,t+1}, ..., 2K_{1,n})$, where $0 \leq \alpha < t$;
(iii) $(3K_{1,1}, 3K_{1,2}, ..., 3K_{1,n})$,

Murugan and Arumugam [59] showed additionally in 1999 that any banana tree where all stars have the same size is graceful by constructing a graceful labelling of these banana trees.

Golomb [30] showed in 1972 that every tree of order $n \leq 5$ is graceful by an exhaustive labelling. Aldred and McKay [5] showed in 1998 that every tree of order $n \leq 27$ is graceful. They employed an algorithm which begins with an arbitrary vertex labelling using distinct labels from $\{0, 1, 2, ..., n - 1\}$ and which switches a pair of labels if the cardinality of the edge label set increases with the switch. Gracefulness is established as the vertex labels range over all permutations until a graceful labelling is found. To illustrate the scale of their computations, the authors note that there are 279,793,450 trees of order 26 and 751,065,460 of order 27.

A tree is said to be 0-*rotatable* (or simply rotatable) if any vertex in the tree may be labelled 0 in some graceful labelling of the tree. Rosa noted the problem of

0-rotatability in graceful trees in 1966 and announced that paths are 0-rotatable. He proved this in a short note in 1977 [65], along with the following:

Theorem 10. *There exists a bipartite labelling of any path in which any vertex may be labelled 0 if and only if the vertex is not the central vertex of P_5.*

Let C be a caterpillar and let $v_1, ..., v_n$ be a longest path in C. Chung and Hwang [25] define the *head* and *tail* of C to be the vertices v_1 and v_n and the *feet* of C to be the internal vertices v_1,...,v_{n-1}. If every foot has the same degree $t + 2$, C is called a t-toed caterpillar. The authors showed in 1981 that every t-toed caterpillar is 0-rotatable, using several modifications of Rosa's construction of a bipartite labelling for arbitrary caterpillars. Note that caterpillars are not in general 0-rotatable: consider the spider $S(1, 1, 3)$ obtained from $K_{1,3}$ by joining one of its leaves v to a vertex of K_2. If v is labelled 0, there is no way to extend it to a graceful labelling.

Van Bussel [75] defines a tree to be *k-centred graceful* if it has a graceful labelling such that the label k is assigned to one of the tree's central vertices. Similarly, a tree T is said to be *k-ubiquitously graceful* if for every vertex there is some graceful labelling of T that assigns that vertex the label k. We call these trees *k-rotatable* to be consistent with previous terminology. Van Bussel defines a *branch* of a diameter-4 tree to be a vertex v adjacent to the centre plus any leaves adjacent to v. He further defines a small subset $\mathcal{D}$ of trees that are not 0-centred graceful by placing conditions on the relative sizes of the two branches in trees of diameter four having exactly two branches. He also shows that all trees of diameter at most four that are not in $\mathcal{D}$ are 0-centred graceful. Van Bussel defines a larger class of trees $\mathcal{D}'$ as the class of trees obtained by identifying a leaf of an arbitrary path with the centre of a tree in $\mathcal{D}$ (note that $\mathcal{D}$ is trivially contained in $\mathcal{D}'$). $\mathcal{D}'$ is shown to be a set of non-0-rotatable trees; furthermore all trees of diameter at most four not in $\mathcal{D}'$ are shown to be 0-rotatable. Van Bussel conjectures, based on computational results, that these results hold in general, that is:

Conjecture 8. *The set of trees which are not 0-centred graceful is exactly the class D.*

Conjecture 9. *The set of trees which are not 0-rotatable is exactly the class D'.*

Two new results on 0-centred graceful labellings are also due to Van Bussel:

Theorem 11.

1. *All trees of diameter four and centre degree ≥ 3 have a 0-centred graceful labelling.*
2. *For k odd, every diameter four tree with k branches has a 0-centred graceful labelling.*

Huang, Kotzig and Rosa [38] believe that an inductive proof of the existence of a graceful labelling should involve combining graceful trees not admitting bipartite labellings with those admitting bipartite labellings. In their 1982 paper, they present several classes of graceful trees without bipartite labellings towards this end. Given that trees with exactly two leaves (namely paths) have bipartite labellings [65], the authors proceeded to study trees with exactly three leaves, namely the spiders $S(p, q, r)$. The authors note that since every tree of this form has a unique vertex of degree three, say u, such a tree can be described by the triple (p, q, r) of distances from u to the leaves. Here we will denote these trees using the spider notation $S(p, q, r)$ for the sake of consistency. Huang et al. proved the following:

Theorem 12.

1. *The tree $S(p, q, r)$ with three leaves has a bipartite labelling if and only if $(p, q, r) \neq (2, 2, 2)$.*
2. *Every tree $S(p, q, r)$ with three leaves has a graceful labelling.*

A tree with exactly four leaves is either a spider $S(p, q, r, s)$ with branch vertex u or a tree with exactly two vertices u, v of degree three. The latter tree is described by $(p, q; r; s, t)$, where the numbers provide the distances between u and the two leaves not separated from u by v, the distance between u and v and finally the distances between v and the two leaves not separated by from v by u. Huang et al. proved the following:

Proposition 1.

1. *If at least two of p, q, r, s do not equal 2 then there exists a bipartite labelling of $S(p, q, r, s)$.*
2. *Every spider $S(p, q, r, s)$ has a graceful labelling.*
3. *Every tree $(p, q; r; s, t)$ has a graceful labelling.*

These results are summarized as follows:

Theorem 13. *A tree with fewer than five leaves is graceful.*

Each tree of diameter three on $r + s + 2$ vertices, denoted T_{rs}, has one vertex of degree $r + 1$ adjacent to one vertex of degree $s + 1$ and $r + s$ leaves. This tree is the spider $S(1, 1, \ldots, 1, 2, 2, \ldots, 2)$ with r 1's and s 2's. Let P_{rs} be the tree of diameter six obtained by replacing each edge in T_{rs} with a path of length two. Huang et al. proved the following about P_{rs}:

Theorem 14.

1. *The tree P_{rs} has a bipartite labelling if and only if $|r - s| \leq 1$.*
2. *Every tree P_{rs} has a graceful labelling.*

A more general nonexistence theorem for bipartite labellings due to the same authors is:

Theorem 15. *Let T be a tree all of whose vertices are of odd degree. Let T^* be obtained from T by replacing every edge of T by a path of length two. If $|V(T)| \equiv 0 \pmod 4$, then T^* does not have a bipartite labelling.*

Denote by $S(4)$ the class of trees of diameter four and partition this class into three subclasses: $S(4) = S(0, 4) \cup S(1, 4) \cup S(2, 4)$ where $S(0, 4)$ contains only the path P_5 of length 4, $S(1, 4)$ contains caterpillars of diameter four and $S(2, 4)$ contains all other trees of diameter four. By showing first that if $T \in S(0, 4) \cup S(1, 4)$ and x is the central vertex of T then there is no bipartite labelling f of T such that $f(x) = 0$, Huang et al. showed the following concerning trees of $S(2, 4)$:

Proposition 2. *If $T \in S(2, 4)$, then T has no bipartite labelling.*

Let $T(q_1, q_2, q_3; s)$ be the tree rooted at its centre x, where x is adjacent to s leaves and to three vertices having degrees $q_1 + 1$,$q_2 + 1$ and $q_3 + 1$, where each of these three vertices are adjacent only to x and leaves of T. In this case, the authors proved the following:

Proposition 3. *For any $q_1, q_2, q_3 \geq 1$ there exists a graceful labelling f of $T(q_1, q_2, q_3; 0)$ with $f(x) = 0$.*

In 1973 Kotzig [48] extended the work of Rosa on balanced trees:

Theorem 16.
1. *If a leaf of a long-enough path is joined to any leaf of an arbitrary tree, the resulting tree is graceful.*
2. *If a long-enough path replaces an arbitrary edge in an arbitrary tree, the resulting tree is graceful.*

Regular bamboo trees are rooted trees consisting of *branches* (paths from the root to the leaves) of equal length, the leaves of which are identified with leaves of stars of equal size. These were shown to be graceful by C. Sekar in 2002 (as referenced in [29]). Olive trees T_k are rooted trees with k branches, the ith branch of which is a path of length i. These are identically the spiders $S(1,..,k)$. Abhyankar and Bhat-Nayak [1] gave direct graceful labelling methods for T_{2n+1} and T_{2n}. Both methods involve assigning labels $q=(n+1)(2n+1)$ or n to the branch vertex (referred to as the apex or root by the authors) of the trees T_{2n+1} and T_{2n} respectively and then assigning labels to the vertices on the k paths adjacent to the root depending on the parity of the path label and the tree in question. In a final step, labels are assigned to the remaining vertices of the tree so that the sum of any two adjacent vertices is either $q-1$ or q in the case of T_{2n+1}, or q or $q+1$ in the case of T_{2n}.

Poljak and Sura [61] showed that all equidescendent c-symmetrical trees are graceful. A tree is *equidescendent* if every two siblings in the tree, none of which are leaves, have the same total number of descendants. Let C be a caterpillar with root v_0 containing a path $v_0,...,v_p$ such that all other vertices of C are leaves. Let $T_0,...,T_p$ be a collection of trees with roots $u_0,...,u_p$. Denote by ${T_0}^*,...,{T_p}^*$ disjoint isomorphic copies of the trees $T_0,...,T_p$. For each vertex $x \in T_i$, let x^* represent the vertex similar to x under the isomorphism. A *c-symmetrical* tree is one resulting from the identifications $v_i = u_i = {u_i}^*$, so that the new tree is *symmetrical with respect to a caterpillar*. The authors also extended their method to the construction of a small class of graceful lobsters obtainable from caterpillars. Poljak and Sura showed the following result:

Theorem 17. *Let T be a gracefully labelled tree. Let G_1 and G_2 be two subgraphs of T each isomorphic to a star $K_{1,rs}$ and labelled with bipartite labellings $\{a\}$,$\{b+1,...,b+sr\}$ and $\{b\}$,$\{a+1,...,a+sr\}$, where $a+sr<b$. Let G_1' and G_2' be the trees with the following structure:*
1. *A root with s direct descendants, each of which have $r-1$ additional descendants;*
2. *Labels of the three levels above are respectively $\{a\}$, $\{b+kr : 1 \le k \le s\}$, $\{a+(s-k)r+1,...,a+(s-k+1)r-1\}$ according to the value of k of the previous level for G_1', and $\{b\}$, $\{a+kr : 1 \le k \le s\}$, $\{b+(s-k)r+1,...,b+(s-k+1)r-1\}$ for G_2'.*

Then a tree T' obtained from T by replacing G_1 and G_2 with G_1' and G_2' respectively is graceful.

Applying this proposition to pairs of adjacent vertices of gracefully labelled caterpillars produces new graceful lobsters.

A tree T rooted at v is *symmetrical* if all the vertices of T in the same level have the same degree. Poljak and Sura [61] show in particular that caterpillars and symmetrical trees, including complete m-ary trees, are graceful. Koh, Rogers and Tan [43] also show that complete m-ary trees are graceful by starting with the gracefulness of stars and applying an inductive method to generate larger graceful trees from smaller ones.

Rosa established that trees of diameter at most three are graceful by gracefully labelling caterpillars [65]. Trees of diameter three are double stars (two stars joined at the central vertices) and trees of lower diameter are stars or K_2, all of which are special types of caterpillars. In 1989 Cahit [21] used a *canonic spiral labelling* to show that all trees of diameter four and some of diameter five are graceful. A canonic spiral labelling of a rooted graph is a bipartite labelling with the root assigned the lowest label, which orders vertices canonically by considering their distance from the root and a fixed rotational ordering of the paths from the root to the leaves. In order to study diameter four trees, the following notation has been introduced:
Recall that $S(4)$ is the class of trees of diameter four with $S(4) = S(0,4) \cup S(1,4) \cup S(2,4)$ where $S(0,4)$ contains only the path P_5, $S(1,4)$ contains caterpillars of diameter four and $S(2,4)$ contains all other trees of diameter four. Cahit states the following lemma:

Lemma 2. *Let T be a gracefully labelled tree. Let v be the vertex of T labelled 0. Then the tree T' obtained by joining v to a leaf of $K_{1,k}$ is graceful.*

Combining the lemma and his canonic spiral labelling procedure, Cahit was able to show that all trees of diameter four are graceful. Furthermore, if $T_{r,k}$ is a rooted tree in which all vertices except the root and the leaves are of even degree and where the lengths of all paths from the root to leaves is k, then $T_{r,k}$ has a graceful labelling.

Zhao [85] also showed in 1989 that all trees of diameter four are graceful.
Zhao first defined three sets of vertices in trees of diameter four as follows, where x is the centre of T:
$A = \{u : d(x,u) = 1$, where u is a leaf$\}$
$B = \{v : d(x,v) = 1$, where v is not a leaf$\}$
$C = \{v : d(x,v) = 2\}$.
The proof proceeds by a construction of the labelling using the above vertex sets in the case $A = \phi$. In the case $A \neq \phi$, the following lemma is applied:

Lemma 3. *Let f_1 be an interlaced labelling of a tree T_1 with $f_1(u) = |V(T_1)|$, and let f_2 be a graceful labelling of a tree T_2 with $f_2(v) = |V(T_2)|$, with $V(T_1) \cap V(T_2) = \phi$. Then there exists a graceful labelling of the tree T obtained by identifying the vertex u of T_1 with the vertex v with T_2.*

In 2001 Hrnčiar and Haviar [37] showed that all trees of diameter five are graceful. Their proof relies on two lemmas:

Lemma 4. *Let a tree T with n edges have a graceful labelling f and let $u \in V(T)$ be such that $f(u) = 0$ or $f(u) = n$. Let H be a caterpillar, $V(T) \cap V(H) = \phi$ and let $v \in V(H)$ be a vertex which has either maximum eccentricity or is adjacent to a vertex of maximum eccentricity. If T' is the tree obtained by identifying u and v, then T' is also a graceful tree.*

Let T be a tree and let $uv \in E(T)$. Denote by $T_{u,v}$ the subtree of T induced by $V(T_{u,v}) = \{w \in V(T) : w = u$ or v is on a u-w path$\}$.

Lemma 5. *Let T be a tree with a graceful labelling f and let u be a vertex adjacent to vertices u_1 and u_2. Let T' be the subtree of $T - \{uu_1, uu_2\}$ that contains u, and let $v \in V(T')$, $v \neq u$.*

(a) *If* $u_1 \neq u_2$, $f(u_1) + f(u_2) = f(u) + f(v)$ *and the tree* T'' *is obtained by identifying the vertex* v *of* T' *with the vertex* u *of* T_{u,u_1} *and* T_{u,u_2}*, then* f *is a graceful labelling of the tree* T'' *too.*

(b) *If* $u_1 = u_2$, $2f(u_1) = f(u) + f(v)$ *and* T'' *is a tree obtained by identifying the vertex* v *of* T' *with the vertex* u *of* T_{u,u_1}*, then* f *is a graceful labelling of* T'' *too.*

The tree T'' is said to be obtained from T by a transfer of T_{u,u_1} and T_{u,u_2} from vertex u to vertex v. Hrnčiar and Haviar use two basic types of transfers of end-edges in their proof. Transfers of the *first type* transfer end-edges with leaves labelled $k, ..., k+m$. These can be realised when $f(u) + f(v) = k + (k+m)$. Transfers of the *second type* transfer two sections of end-edges $k, k+1, ..., k+m$ and $l, l+1, ..., l+m$. These transfers can be realised when $f(u) + f(v) = k + l + m$. Note that both restrictions on when transfers are possible follow from Lemma 5.

Let T be a tree of diameter five with central vertices a and b. Let x be a vertex adjacent to a central vertex a but not a central vertex itself (i.e. $x \neq b$). The subtree $T_{a,x}$ is said to be a *branch* (at the vertex a) if it is a subtree of diameter 2. A branch $T_{a,x}$ is an *odd branch* if the degree of the vertex x is *even*; otherwise it is an *odd branch.*

Let T be a tree of diameter five with central vertices a and b. Hrnčiar and Haviar define the following six parameters in order to label T:

- cardinalities of the sets of all odd and even branches at the vertex a;
- cardinalities of the sets of all odd and even branches at the vertex b;
- cardinalities of the set of all end-edges incident with the two vertices.

Proceeding with cases based on the parities of these six sets, the authors first show the following result.

Theorem 18. *Every tree* T *of diameter 5 is graceful or nearly graceful, that is if* T *is of size* $n+1$ *then the cardinality of the edge label set is either* n *or* $n+1$*.*

Finally, the authors define a third main type of transfer called the *backwards double 8-transfer* which involves a sequence of eight transfers of the first type. The main result then follows upon further parity considerations:

Theorem 19. *Every tree of diameter five is graceful.*

In 1979 Bermond conjectured that all lobsters are graceful (as cited in [29]). So far only a few special cases are known. Wang et al. [76] proved the following in 1994:

Theorem 20. *Let* $\{T_i\}$, $i = 1, 2, ..., 2n+1$, *be stars, where the degrees of the central vertices are either all positive even numbers or all positive odd numbers. For positive integers* $a_0, ..., a_m$ *with sum* n*, select any* $2a_0 + 1, 2a_1, ..., 2a_m$ *stars and identify a leaf of each with the* $m+1$ *vertices of a path* $x_0, ..., x_m$*. The lobster obtained in this way is graceful.*

4. **Generating Graceful Trees.** It would be easy to generate every graceful tree of order n if the 0 label in a graceful labelling could be placed at any vertex in every tree of order $n-1$. Recall that a tree with this property is called 0-rotatable. (This is also referred to as 0-ubiquitously graceful.) If this were the case, then we could just add a leaf to a vertex labelled with 0 and label the new vertex with n to get an induced edge labelling of n on the new edge. Unfortunately though, not all trees

are 0-rotatable. Because of this, other methods were created to construct graceful trees.

It is easy to see that if $f(v)$ is a graceful labelling for a tree T of order n, then $\theta(v) = n - f(v)$ is also a graceful labelling for T. The transformation $\theta(v) = n - f(v)$ is sometimes referred to as the *inverse transformation* of the labelling. An elementary construction method for generating a family of graceful trees can be implemented by performing the inverse transformation θ on the graceful labelling of a tree T and then attaching a new vertex to the vertex labelled 0 under θ.

In 1973 Stanton and Zarnke [73] were the first to develop a nontrivial algorithm for constructing graceful trees. Their method became the basis of many construction methods to follow. (It should be noted that Stanton and Zarnke use the convention that vertices are assigned the labels $\{1, \ldots, n\}$.)

The construction is as follows: Take two graceful trees S and T of order n_S and n_T, respectively. Copies of T will be attached to either n_S or $n_S - 1$ of the vertices of S. In order to produce a graceful labelling they outlined the following steps:

1. Label the n_S or $n_S - 1$ copies of T by adding some multiple of n_T to the vertex labels used in the graceful labelling of T. This multiple is chosen by the following algorithm:
   - Select an arbitrary fixed vertex z in the original tree T.
   - Use $L_i(a)$ to designate the label in T_i of the vertex a of T, and define:

   $$L_i(a) = \begin{cases} (r+1-i)n + 1 - L(a), & \text{if } d(a,z) \text{ is odd;} \\ in + 1 - L(a), & \text{if } d(a,z) \text{ is even;} \end{cases}$$

   where $r = n_S$ or $n_S - 1$, depending on how many copies of T are being attached to S. Note that this labelling uses a vertex label set of $\{1, ..., n\}$, but the same edge labels are induced by shifting the vertex labels down by 1, resulting in the vertex label set $\{0, ..., n-1\}$.
2. Relabel the vertices of S by multiplying all the graceful labels by n_T and adding an arbitrary fixed c, where $0 \le c < n_T$.
3. Identify vertices in the relabelled S and the copies of T which have identical vertex labels. This operation of identifying vertices is referred to as graphting. The resulting tree is also a graceful tree.

The tree that results from implementing this construction is classified as either Type I or Type II. A Type I tree results from attaching n_S copies of T to S and a Type II tree results from attaching $n_S - 1$ copies of T to S. (Stanton and Zarnke call the trees resulting from their construction balanced trees, however note that these are not the same as the balanced trees defined in Section 3.) It is interesting to note that the set of all trees derived using a vertex z as a fixed vertex in T is the same as the set of trees derived using a different vertex y (but corresponding trees may not be the same).

In [43] Koh, Rogers and Tan gave a variation of Stanton and Zarnke's construction. This idea led to many more constructions [43], [44],[45], [46], [47]. Let $T(n)$ be a tree of order n, and θ the graceful labelling of $T(n)$. (It should be noted that Koh, Rogers and Tan use the convention that $\theta(V(T)) = \{1, \ldots, n\}$.) Then $(T(n), \theta)$ is called a graceful system. Let w be the unique vertex in $T(n)$ with $\theta(n) = n$. A bigger graceful tree is constructed based on $T(n)$, θ and w. For each integer $p \ge 1$, let $T_1(n), T_2(n), \ldots, T_p(n)$ be p disjoint isomorphic copies of $T(n)$ and for each $i = 1, 2, \ldots, p$, let w_i be the isomorphic image of w in $T_i(n)$. Adjoin to the graph $\cup_{i=1}^{p} T_i(n)$ a new vertex w_0 and p edges $w_0w_1, w_0w_2, \ldots, w_0w_p$. The new tree obtained from this construction is denoted by $T_w^p(n)$.

Using their construction Koh, Rogers and Tan were able to show that, given a graceful system $(T(n), \theta)$ with $\theta(w) = n$ where $w \in T(n)$, then for each integer $p \geq 1$, there exists a graceful valuation θ^* on $T_w^p(n)$ such that $\theta^*(w_0) = np + 1$.

A result of this construction is that for each positive integer m, every complete m-ary tree is graceful. This provided a response to the question posed by Cahit in 1976: Are all complete binary trees graceful?

In 1979 Koh, Rogers and Tan created two new constructions based on their original idea [45]. Using the setup from before, denote by $T_w^p(n)^*$ the tree obtained by identifying $w_1 = w_2 = \ldots = w_p = w$ on the set $\cup_{i=1}^{p} T_i(n)$. If some specified conditions are satisfied for the labels on vertices neighbouring the vertex w, then they showed that there exists a labelling θ^* on $T_w^p(n)^*$ such that the system $(T_w^p(n)^*, \theta^*)$ is graceful.

Let $(T(m), \theta')$ and $(T(n), \theta^*)$ be two given graceful systems where $T(m) = \{w_1, w_2, \ldots, w_m\}$. Let v be an arbitrary fixed vertex in $T(n)$. Based upon the tree $T(m)$, adjoin an isomorphic copy $T_i(n)$ of $T(n)$ to each vertex w_i $(i = 1, 2, \ldots, m)$ by identifying v^* and w_i. All the m copies of $T(n)$ just introduced are pairwise disjoint and no extra edges are added. Such a new tree obtained is denoted by $T(m)\Delta T(n)$. It is obvious that $|T(m)\Delta T(n)| = mn$ and $T(m)\Delta T(n) \not\cong T(n)\Delta T(m)$ in general. Using this construction, if $(T(m), \theta')$ and $(T(n), \theta^*)$ are two graceful systems then there exists a labelling θ on $T(m)\Delta T(n)$ such that the system $(T(m)\Delta T(n), \theta)$ is graceful.

In [44] Koh, Rogers and Tan studied interlaced labellings in order to provide another method of constructing graceful trees. If θ is an interlaced labelling of T then so is the inverse transformation of θ, as well as the labelling θ' where

$$\theta'(v) = \begin{cases} p + 1 - \theta(v) & \text{if } \theta(v) \leq p, \\ n + p + 1 - \theta(v) & \text{if } p < \theta(v). \end{cases}$$

They also gave an algorithm for an interlaced labelling.

During the 1980s Koh, Rogers and Tan extended their previous work and examined the families of trees generated by their constructions. Much of their work during this time focused on the properties of interlaced labellings.

In 1993 the team of Jin, Liu, Lee, Lu and Zhang published two papers [40], [52], each paper giving a new way of constructing graceful trees from smaller graceful trees.

In the first paper Jin et al. gave the concept of the *joint sum* operation. Given two trees T and R, define a new tree by joining a vertex of T with a vertex of R. This tree is called the joint sum of T and R and is denoted by $\langle T + R \rangle$. Note that the joint sum of two trees is not unique. Similarly, we can define $\langle T_1 + T_2 + \ldots + T_n \rangle$. When $T_1, T_2, \ldots, T_n$ are all isomorphic, $\langle T_1 + T_2 + \ldots + T_n \rangle$ is denoted as $\langle nT_1 \rangle$.

Jin et al. [40] proved many results on the gracefulness of trees obtained by performing the joint sum operation.

Theorem 21. *Let T and R be two graceful trees. Then the joint sum $\langle T + 2R \rangle$ by joining the vertices with the graceful label 0 is also graceful.*

This result generalized nicely into the following:

Theorem 22. *If T and R are two graceful trees, then $\langle \lambda T + 2\mu R \rangle$ is also graceful $(\lambda, \mu = 1, 2, \ldots)$.*

Let R^* be a graceful tree with a bipartition (X, Y). If there is a graceful labelling $f(v)$ such that

$$\max_{v \in X} f(v) < \min_{v \in Y} f(v),$$

then R^* is called a *glue tree*. In addition, a vertex v^* which satisfies

$$f(v^*) = \min_{v \in Y} f(v)$$

is called a *joint vertex*; or simply a *joint*; a vertex v_* which satisfies

$$f(v_*) = \max_{v \in X} f(v)$$

is called a *glue vertex*, or simply a *glue*.

This concept of a glue tree was used to prove many properties of the joint sum operation.

Proposition 4. *If R is a graceful tree, then $\langle 2\mu R \rangle$ is a glue tree ($\mu = 1, 2, ...$).*

Theorem 23. *Let T be a graceful tree and R^* a glue tree. Then the joint sum $\langle T + R^* \rangle$ by joining the vertex of T with graceful label 0 with a joint vertex of R^* is also graceful.*

Corollary 1. *If T is a graceful tree and R^* is a glue tree, then $\langle \lambda T + \mu R^* \rangle$ is a graceful tree ($\lambda, \mu = 1, 2, \ldots$).*

In their second paper, Jin et al. [52] used the glue tree definition to give the *radical product* operation. Given two trees T and R, specify for each tree a vertex as the root. By gluing the two roots we obtain a new tree, called the *radical product* of T and R, denoted by $\langle T \bullet R \rangle$. When T and R are isomorphic, $\langle T \bullet R \rangle$ is simply denoted as $\langle R^2 \rangle$. Similarly, we define $\langle T_1 \bullet T_2 \bullet \ldots \bullet T_n \rangle$ and $\langle T^n \rangle$.

As in the first paper, many properties of this new operation were proved.

Theorem 24. *Let T and R be two graceful trees with the same order. If for each tree there is a leaf with graceful label 0, then the radical product $\langle T^m \bullet R^{2n} \rangle$ using these two vertices as roots is a graceful tree, where m and n are positive integers.*

The above result was generalized to a list of trees:

Theorem 25. *Let T_i $(i = 1, 2, \ldots, k)$ be graceful trees with q edges. If for each tree T_i there is a leaf b_i with graceful label 0, then the radical product $\langle T_1^{m1} \bullet T_2^{2m_2} \bullet \ldots T_k^{2m_k} \rangle$ obtained by using b_i as the root is also graceful, where m_i $(i = 1, 2, \ldots, k)$ are positive integers.*

Theorem 26. *If T^* is a glue tree and R is a graceful tree, then the radical product $\langle T^* \bullet R \rangle$ is also graceful, where the root of T^* is the glue vertex and the root of R is the vertex with graceful label 0.*

If we have a tree T of size p with graceful labelling $f(v)$ then we can form the tree T' by joining a new vertex w to the vertex that is labelled 0. Let w be labelled $p + 1$. Then the labelling f' defined by

$$\begin{aligned} f'(u) &= f(u) \quad \text{if } u \in V(T) \\ f'(w) &= p + 1 \end{aligned}$$

is a graceful labelling for T'. Using the inverse transformation of $f'(v)$ we can obtain a graceful labelling of the new tree that has the 0 label on a leaf. Then the construction given above can be used to generate a larger graceful tree. This method can be iterated to produce a whole family of graceful trees.

In 1998, Burzio and Ferrarese [20] gave a generalization of Koh, Tan and Roger's method to construct $T(m)\Delta T(n)$. If each edge $e = uv$ of a graph G is replaced by a new vertex w and the edges uw and wv, then the resulting graph is called the *subdivision graph of* G. Burzio and Ferrarese used this generalized construction to show that the subdivision graph of a graceful tree is a graceful tree. (In general this cannot be used to show that the subdivision graph of any graceful graph is graceful. A counterexample is the cycles. For example, C_3 is graceful but C_6 is not.)

In 2003 the concept of a Skolem sequence was used to generate new graceful trees [58]. A *Skolem sequence* of order n is a sequence $(s_0, s_1, \ldots, s_{2n-1})$, which has the following properties:

1. Its entries are taken from the set $\{1, 2, \ldots, n\}$.
2. For all $k \in \{1, 2, \ldots, n\}$, there are exactly two subscripts $i(k)$ and $j(k)$, for which $s_{i(k)} = s_{j(k)} = k$.
3. For all $k \in \{1, 2, \ldots, n\}$, $|i(k) - j(k)| = k$.

A *hooked-Skolem sequence* $(s_0, s_1, \ldots, s_{2n})$ of order n is defined by the same properties as a Skolem sequence with the additional property that $s_{2n-1} = 0$. The zero is referred to as a hook.

A Skolem sequence of order n exists if and only if $n \equiv 0, 1 \pmod 4$. A hooked-Skolem sequence of order n exists if and only if $n \equiv 2, 3 \pmod 4$. A construction for Skolem sequences and hooked-Skolem sequences of order n can be found in [70].

Morgan and Rees [58] gave a method of constructing graceful trees from a Skolem or a hooked-Skolem sequence. This construction gives a method to generate graceful lobsters.

5. **Graceful Graphs and an Equivalent to the Graceful Tree Conjecture.** A tree T of order n is said to be *strongly graceful* if T contains a perfect matching M and T admits a graceful labelling f such that $f(u) + f(v) = n - 1$ for every edge $uv \in M$. Broersma and Hoede [19] prove that the following Strongly Graceful Tree Conjecture is equivalent to the Graceful Tree Conjecture:

Conjecture 10. *Every tree containing a perfect matching is strongly graceful.*

Let the *corona*, *cor*(T), (called a *spiketree* by Broersma and Hoede) of a tree T of order n be obtained by adding n new vertices to T along with n edges matching the old and new vertices. The *contree* of a tree T with a perfect matching M is obtained from T by contracting the edges of M. The equivalence of Conjecture 10 and the Graceful Tree Conjecture is established by first using a graceful labelling of the contree of a tree with a perfect matching to show that a tree with a perfect matching is strongly graceful. This amounts to finding vertex labels that sum to $n - 1$ and have absolute difference as prescribed by double the edge label in the original graceful labelling. The converse is established by taking the corona S of an arbitrary tree T on n vertices. S is strongly graceful by assumption, and has order $2n - 1$. Define the two sets matched by the perfect matching to be those containing odd and even labels; by giving labels to vertices of T according to half the even vertex label of the corresponding edge in the strong graceful labelling of S, a graceful labelling is constructed.

The following lemma is also due to Broersma and Hoede:

Lemma 6. *Let T be a tree containing a perfect matching and let T^c be the contree of T. Then T is strongly graceful if and only if the corona T^* with contree T^c is strongly graceful.*

As the authors note, this shows that in order to prove the Graceful Tree Conjecture, it would be sufficient to show that every corona is strongly graceful. However, as the authors observe, since a strongly graceful labelling of a corona immediately yields a graceful labelling of its contree, this is hardly an improvement. Attempts to prove the Strongly Graceful Tree Conjecture have been frustrated by the fact that the label 0 cannot be assigned to an arbitrary vertex, the same stumbling block that arose in attempts to prove the original Graceful Tree Conjecture.

A useful result on strongly graceful labellings is the following [19]:

Theorem 27. *Every tree containing a perfect matching and having a caterpillar as its contree is strongly graceful.*

Note that trees meeting the hypothesis above can be used to generate new strongly graceful trees with the original tree as their contrees, by successively taking the coronas of a sequence of trees. This procedure generates what the authors term strongly graceful *long-legged caterpillars.*

Graham and Sloane [32] established that almost all graphs are not graceful. They also state some asymptotic results on the sizes of extremal graphs on n vertices. Let $g(v)$ denote the number of edges in the graceful graph on n vertices with largest size. It is known that $\lim_{v\to\infty} \frac{g(v)}{v^2}$ exists and satisfies

$$\frac{1}{3} \leq \lim_{v\to\infty} \frac{g(v)}{v^2} \leq 0.411$$

Golomb [30] proved the original result on complete graphs in 1972:

Theorem 28. *If $n > 4$ the complete graph K_n is not graceful.*

The proof relies on the fact that a graceful labelling requires the assignment of labels 0 and e to two vertices in order to achieve the edge label e. The argument proceeds by requiring that the label 1 be assigned to another vertex (or $e-1$ in the symmetrical case). Continuing in this fashion, labels $0, 1, 4, e-2, e$ are assigned to some vertices. For K_4 with $e = 6$, this constructs a graceful labelling. However, there is no way to obtain an edge labelled $e-5$ because each of the ways to obtain $e-5$ as a difference of two numbers requires an impossible vertex label. Thus K_n is not graceful whenever $e-5 > 4$, or when $n \geq 5$.

The following is due to Golomb [30]:

Theorem 29. *For all positive integers a and b, the complete bipartite graph $K_{a,b}$ is graceful.*

Assigning the vertices in the partite set of cardinality a the labels $0, ..., a-1$ and the vertices in the other partite set the labels $a, 2a, ..., ba$ exhibits a graceful labelling.

Rosa [64] showed the following results on graceful cycles in 1967:

Theorem 30. *A bipartite labelling of C_n exists if $n \equiv 0 \pmod 4$.*

Proof: Label the vertices v_i of C_{4k} with label a_i as follows, where the v_i's are arranged cyclically:

$$a_i = \begin{cases} (i-1)/2, & \text{i odd} \\ n+1-i/2, & \text{i even, } i \leq n/2 \\ n-i/2, & \text{i even, } i > n/2. \end{cases}$$

□

Theorem 31. *A graceful labelling of C_n exists if $n \equiv 3 \pmod 4$.*

Proof: Label the vertices v_i of C_{4k+3} with label a_i as follows, where the v_i's are arranged cyclically:

$$a_i = \begin{cases} n+1-i/2, & \text{i even} \\ (i-1)/2, & \text{i odd, } i \leq (n-1)/2 \\ (i+1)/2, & \text{i odd, } i > (n-1)/2 \end{cases}$$

□

If $n \equiv 1, 2 \pmod 4$ then C_n does not admit a graceful labelling. In fact, Rosa [64] proves the following more general result:

Lemma 7. *If G is an Eulerian graph with n edges where $n \equiv 1, 2 \pmod 4$, then there does not exist a graceful labelling of the graph G.*

Note that the graceful labelling of cycles is not necessarily unique. We construct two non-isomorphic and non-inverse graceful labellings of both C_7 and C_8:

Denote the cycle C_7 by $v_1v_2\cdots v_7$ oriented cyclically. The two labellings are: $(v_1, \ldots, v_7) = (0, 7, 1, 6, 3, 5, 4)$ and $(v_1, \ldots, v_7) = (0, 7, 1, 6, 2, 4, 3)$.

Denote the cycle C_8 by $v_1v_2\cdots v_8$ oriented cyclically. The two labellings are: $(v_1, \ldots, v_8) = (0, 8, 1, 7, 2, 5, 3, 4)$ and $(v_1, \ldots, v_8) = (0, 7, 1, 5, 2, 4, 3, 8)$.

Hebbare [35] conjectured in 1976 that all *wheels* W_n are graceful, where $W_n = K_n + K_1$. Hoede and Kuiper (as referenced in [29]) proved this in 1987. Hebbare did early work on labellings of wheels, briefly listing all graceful labellings of W_n, $n = 3, 4, 5$ and those for W_6 where the centre is labelled 0. This work led him to consider other non-Eulerian graphs, and to formulate the following conjecture [35]:

Conjecture 11. *A non-Eulerian graph with at least two blocks, each block being a complete graph on at least three vertices, is not graceful.*

The graph $C_m + \overline{K_n}$, called the *join* of C_m and $\overline{K_n}$, is obtained by taking the union of C_m with $\overline{K_n}$ and joining all pairs of vertices that appear in different graphs. These are generalized wheels called *n-cones* and are denoted by Bhat-Nayak and Selvam [13] as $C_m \vee K_n^c$. The gracefulness of n-cones is still in general an open problem. However, Bhat-Nayak and Selvam have established the following:

Theorem 32. *$C_m + \overline{K_n}$ is graceful for any $n \geq 1$ and $m \equiv 0, 3 \pmod{12}$.*

Bhat-Nayak and Selvam defined a *special* labelling of C_n as one assigning vertices distinct labels from $1, ..., 2n$ such that:

1. For each $1 \leq i \leq n$, there exists a vertex of C_n labelled with either $2i - 1$ or $2i$.
2. The set of induced edge labels is the complement of the set of vertex labels in the set $\{1, ..., 2n\}$.
3. If an odd and even label are assigned to adjacent vertices, the odd label is less than the even one.

The authors proved the following result about special labellings:

Theorem 33. *If a graph G on m vertices with m edges has a special labelling, then the graph $G + \overline{K_n}$ is graceful for $m \geq 3$ and $n \geq 1$.*

By treating the four cases C_{24m}, C_{24m+3}, C_{24m-9},C_{24m-12} separately, and constructing special labellings in each case, the authors established the result stated in Theorem 32.

Additionally, Bhat-Nayak and Selvam have shown the following 2-cones and n-cones to be graceful:

$C_5 + \overline{K_2}$ and $C_9 + \overline{K_2}$; $C_4 + \overline{K_n}$, $C_7 + \overline{K_n}$, $C_{11} + \overline{K_n}$, $C_{19} + \overline{K_n}$.

Rosa's [64] parity condition for graceful graphs of size $m \equiv 1, 2 \pmod 4$ states that any graph with all vertices of even degree is not graceful. Using this condition, Bhat-Nayak and Selvam deduce the following:

Theorem 34. *$C_m + \overline{K_n}$ is not graceful for n even and $m \equiv 2, 6, 10 \pmod{12}$.*

Fu and Wu [28] showed the following result on graceful graphs in 1990:

Theorem 35. *If T is a tree with a graceful labelling and S_n is a star on n vertices then $T + S_n$ is graceful.*

A second result of Fu and Wu [28] pertains to the Cartesian product of trees with bipartite labellings:

Theorem 36. *Let T be a tree with partite sets of size n_1 and n_2 where $|n_1 - n_2| \leq 1$. Then, for $m \geq 2$, the Cartesian product $T \times P_m$ has a bipartite labelling.*

A corollary of the above is that the *net* $P_m \times P_n$ has a bipartite labelling for all positive integers m and n.

Bodendiek conjectured that every graph consisting of a cycle plus a chord is graceful (as referenced in [86]), and various proofs have established this result, including one in 1986 by Chen and Zhi Zeng. In more recent work, Zhi Zeng [86] has proved a generalization of the Bodendiek conjecture:

Theorem 37. *Apart from four exceptional cases, graphs consisting of three independent paths joining two vertices are graceful.*

The four exceptional cases are phrased in terms of the parities of the three path lengths and their linear combinations, as well as conditions on the relative lengths of the three paths. Zhi Zeng's method uses the graceful labelling (GL)-matrix to study the behaviour of labellings on such graphs:

The L-matrix of a graph G with respect to a vertex labelling ϕ consists of the adjacency matrix of G, where the rows and columns are canonically ordered according to the order $0, ..., |E|$ corresponding to the label of the appropriate vertex. Of course, there may be numbers in this list that are not used as labels, and in this case the row and column are simply zero vectors. In the upper triangular portion of the matrix, there are altogether $|E|$ oblique lines which parallel the main diagonal. If there is only one edge index 1 in each oblique line, then the matrix is called a GL-matrix, representing the fact that ϕ is a graceful labelling of G.

6. **Variations on Graceful Labellings.** Let G be a graph with $|V(G)| = p$, $|E(G)| = q$, and (l, l^*) be a function pair mapping the vertices and edges into the set of integers. Define G as *edge-graceful* if there is a function pair (l, l^*) such that l is onto $\{0, 1, \ldots, p-1\}$, l^* is onto $\{1, 2, \ldots, q\}$, and for $u \in V(G)$

$$l(u) = \sum_{uv \in E(G)} l^*(uv) \pmod p.$$

That is, take the sum of the labels of all edges incident with u and reduce modulo p.

It is conjectured that all trees of odd order are edge-graceful. This is known as **Lee's Conjecture**. In [71] Simoson showed that all three-legged spiders of odd order are edge-graceful, and that all four-legged spiders of odd order are edge-graceful. He also showed how to inductively attach appropriately long, new legs onto spiders so that the new spider is edge-graceful.

There is a related concept of *super-edge-graceful graphs*. Let

$$P = \begin{cases} \pm 1, \dots, \pm \frac{p}{2}, & \text{if } p \text{ is even,} \\ 0, \pm 1, \dots, \pm \frac{p-1}{2}, & \text{if } p \text{ is odd.} \end{cases} \quad Q = \begin{cases} \pm 1, \dots, \pm \frac{q}{2}, & \text{if } q \text{ is even,} \\ 0, \pm 1, \dots, \pm \frac{q-1}{2}, & \text{if } q \text{ is odd.} \end{cases}$$

Then a graph G is *super-edge-graceful* if there is a function pair (l, l^*) such that l is onto P, l^* is onto Q and

$$l(u) = \sum_{uv \in E(G)} l^*(uv).$$

It has been shown [57] that if G is a tree of odd order and is super-edge-graceful, then G is edge graceful. Simoson used this fact in [72] to construct new types of edge-graceful trees. It should be noted though that super-edge-graceful graphs are known to exist that are not edge-graceful. Mitchem and Simoson also used super-edge-graceful trees in [57] to construct various new edge-graceful graphs. Their constructions also gave a proof to show that all spiders having legs the same length are edge-graceful.

A connected graph G of order n and size m with $m \geq n$ is *harmonious* if there exists a labelling $\phi : V(G) \to \mathbb{Z}_m$ of the vertices of G with distinct elements $0, 1, \dots, m-1$ of $\mathbb{Z}_m$ such that each edge uv of G is labelled $\phi(u) + \phi(v)$ (addition in $\mathbb{Z}_m$) and the resulting edge labels are distinct. Such a labelling is called a *harmonious labelling*. If G is a tree then exactly two vertices are labelled with identical labels; otherwise the definition is the same. Harmonious labellings of graphs were studied by Graham and Sloane in [32] in the context of problems stemming from error-correcting codes. The following is known as the **Graham-Sloane Conjecture**:

Conjecture 12. *Every nontrivial tree is harmonious.*

Aldred and McKay showed in [5] that all trees on at most 27 vertices are harmonious.

For a labelling f of a graph G with induced edge labelling $\overline{f}$, denote by $v_f(i)$ (respectively $e_{\overline{f}}(i)$) the number of vertices (respectively edges) with the label $i \in \{0, 1, \dots, k\}$, $k \leq |E(G)|$. The labelling f is called $(k+1)$-equitable if we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_{\overline{f}}(i) - e_{\overline{f}}(j)| \leq 1$, for $i \neq j$, $i, j = 0, 1, \dots, k$. For $k = 1$, a 2-equitable labelling has been given the name *cordial labelling*. For $k = |E(G)|$, the $(|E(G)| + 1)$-equitable labelling is exactly the same as a graceful labelling of G. In [21] Cahit states a previous result of his own that every tree is cordial; however, all attempts at generalizing this result to equitable labellings have failed.

A graceful labelling f of a tree is said to be *ordered graceful* if for every three vertices x, y and z of the tree T with $xy, yz \in E(T)$ we have either

$$\begin{array}{lll} f(x) < f(y) & \textit{and} & f(y) > f(z), \quad \textit{or} \\ f(x) > f(y) & \textit{and} & f(y) < f(z). \end{array}$$

If a graph has a bipartite labelling then it is also ordered graceful. In [22] Cahit gives an ordered graceful labelling of spiders with all legs of length 2 on up to 23 vertices.

Consider a graph G of order p and size q. The graph G is said to be (k,d)-graceful, where k and d are positive integers, if the p vertices admits an assignment of a labelling of numbers $0,1,2,\dots,k+(q-1)d$ such that the values on the edges defined as the absolute difference of the labels of their end vertices form the set $\{k,k+d,\dots,k+(q-1)d\}$. In [36] Hegde and Shetty constructed a class of trees called T_P-trees that they showed to be (k,d)-graceful for all positive integers k and d. They also showed that if T is a T_P-tree then the subdivision graph of T is also (k,d)-graceful for all positive integers k and d.

Let $H = H(A,B)$ be a bipartite graph with size n and partite sets A and B. A pair of injective maps $f_A : A \to \{0,\dots,n-1\}$ and $f_B : B \to \{0,\dots,n-1\}$ is a *bigraceful labelling* of H if the induced labelling on the edges $f_e : E(H) \to \{0,\dots n-1\}$ defined by $f_e(xy) = f_B(y) - f_A(x)$ is also injective. A bigraceful labelling f of $H(A,B)$ is said to be *consecutive* if $f(A) = \{0,\dots,|A|-1\}$ and $f(B) = \{|A|-1,\dots,n-1\}$. Then the labelling $\overline{f} : V(H) \to \{0,\dots,n\}$ defined as $\overline{f}(A) = f(A)$ and $\overline{f}(B) = f(B)+1$ is the same as a bipartite labelling. In [53] Lladó and López used properties of bigraceful labellings to construct bigraceful graphs. These constructions yield graphs that decompose $K_{n,n}$, and thus give results towards verifying Häggkvist's Conjecture.

7. **Asymptotics and Bounds for Graceful Trees.** Abrhám and Kotzig showed in 1990 that the number of bipartite labellings of paths of order n grows asymptotically at least as fast as 1.3953^n (as referenced in [6]). Aldred et al. [6] improved this result to $(\frac{5}{3})^n$ by combining two bipartite labellings as follows. A *k-pendant* bipartite labelling is a bipartite graceful labelling of a path P_n in which the vertex label k is assigned to a leaf. By combining k-pendant bipartite labellings of P_n and P_{2t}, Aldred et al. showed that inserting a new edge joining the vertex labelled k in P_n to the leaf not labelled k in P_{2t} produced a k-pendant bipartite labelling of P_{n+2t}.

Let $\tau_k(n)$ denote the number of inequivalent (neither isomorphic nor inverse) k-pendant bipartite labellings of the path P_n. The original results by Abrhám and Kotzig used the recurrence $\tau_k(n+2t) \geq \tau_k(n)\tau_k(2t)$, with $\tau_1(n) = 1$. Aldred et al. defined $b_{k,t} = (\tau_k(2t))^{\frac{1}{2t}}$ and $c_{k,t}$ to be the minimum of $\frac{\tau_k(n)}{(b_{k,t})^n}$ taken over all n such that $k+1 \leq n \leq k+2t$. By showing that $\tau_k(n) \geq c_{k,t}(b_{k,t})^n$ for $n,t > k > 1$, and by determining values of $b_{k,t}$ for various pairs k,t, Aldred et al. were able to improve the previous bound. The authors note that their result has an application in topological graph theory. The *current assignment technique* allows the construction of surface embeddings of large graphs from smaller graphs edge-labelled with certain elements of finite groups. Graceful labellings of paths in this context induce current assignments on certain cubic graphs generating vertex-transitive triangular embeddings of complete graphs. Thus improved lower bounds on the number of graceful labellings of paths may lead to improved bounds on the number of vertex-transitive triangulations of complete graphs. The authors additionally raise the question of whether $b_{k,t}$ tends to infinity as t does. If so, the number of vertex-transitive triangulations of surfaces by complete graphs may have super-exponential growth [6].

Canonical adjacency matrices were seen in the end of Section 5. Using adjacency matrices of graceful trees, Bloom [14] showed that there are exactly $n!$ graceful labellings of graceful graphs with n edges and $n+1$ vertices. Rosa characterizes canonical adjacency matrices as those having 1's in each row and column and containing exactly one 1 on each diagonal other than the main diagonal. He also points out that if this characterization of the adjacency matrices of graceful trees could be used to enumerate graceful trees, it would be possible to attack the Graceful Tree Conjecture in this way as the total number of trees is known. Since the number of graceful labellings is varied and is not well-known for arbitrary trees, this line of attack has not proved fruitful. Despite caterpillars having already been determined graceful, Bloom [14] has successfully enumerated caterpillars on n edges:

$$C(n) = 2^{n-3} + 2^{\lceil (n-4)/2 \rceil}.$$

In 1995 Rosa and Širáň [66] defined the *α-size* of a tree T to be the maximum number of distinct values of the induced edge labels taken over all bipartite labellings of T. Let $\alpha(n)$ be the smallest α-size among all trees with n edges. The authors show the following result on $S(2,2,...,2)$ spiders, (referred to as *m-comets* by the authors), which are stars with size m where each edge has been subdivided once:

Theorem 38. *The α-size of any m-comet with size $2m$ is equal to $\lfloor \frac{10m+4}{6} \rfloor$.*

For arbitrary trees, the authors showed that $\alpha(n) \leq \lfloor \frac{5n+9}{6} \rfloor$. They believe that the constant $5/6$ is asymptotically the best possible and that $\alpha(n) \sim 5n/6$. A lower bound on the α-size of arbitrary trees is:

Theorem 39. *Let T be an arbitrary tree with size $n \geq 3$. Then the α-size of T is at least $\lfloor 5(n+1)/7 \rfloor$.*

As a consequence of this the *gracesize*, or the maximum number of distinct edge labels induced by a graceful labelling, is also bounded below by $\lfloor 5(n+1)/7 \rfloor$ for arbitrary trees.

In 1998 Bonnington and Širáň investigated bipartite labellings of trees with maximum degree three [17]. Let $\alpha_3(n)$ be the smallest α-size among all trees with n vertices and maximum degree three. The authors showed that $\alpha_3(n) \geq 5n/6$ for all $n \geq 12$. This also shows that every tree on $n \geq 12$ vertices and with maximum degree three has gracesize at least $5n/6$. Using a computer search, the authors also established that $\alpha_3(n) \geq n-2$ for all $n \leq 17$. The methods used in the second paper are essentially the same as in the original, employing an inductive argument based on the removal of an edge e of a tree T such that the two components of $T-e$ have at least six vertices.

8. **Summary.** Current decomposition work is focused more on using bipartite labellings towards a proof of Häggkvist's Conjecture than on direct attacks on Ringel's Conjecture. There has been a recent emphasis on resolvable designs in an attempt to restrict the scope of decomposition problems. The classes of graceful trees and graphs that are being generated by new operations such as those mentioned in Sections 3 and 4 tend to be small (though still infinite), meaning that overall progress in constructing support for the Graceful Tree Conjecture has been slow. Much attention has been directed towards work using variations of graceful labellings, however problems encountered in trying to solve the Graceful Tree Conjecture reappear when working with other labellings. Constructions of classes of graphs are still only possible in special cases, and methods of proof do not generalize much better to

arbitrary trees when other labellings are considered. Because of these difficulties many new conjectures have been posed for other labellings and work towards the Graceful Tree Conjecture has become unfocused.

Our reasons for writing this survey paper were several. No current broad survey of what is known about graceful trees previously existed in English. We found, in surveying the literature, that there was sometimes confusion about whether a particular class of trees (e.g. lobsters) is known to be graceful, and we found it useful to compile all such information in one place. Terminology for various classes of graphs, operations, and labellings is far from standardized and we have strived to present the most conventional in each instance and also to mention the variations occurring in the literature. Our survey is not as comprehensive as the Electronic Journal of Combinatorics' Dynamic Survey by Gallian, but is much more focused on results on graceful trees than those on graphs in general.

A Note on the References. We have listed below the 51 papers cited in our work, along with an additional 35 papers. The additional references reflect ten sources that were not directly cited here, but that may be of interest to readers, as well as 25 more recent publications that were not available when we began our survey.

Acknowledgements. The authors would like to thank Dr. C. M. Mynhardt for suggesting this problem as a reading topic. We are grateful for many helpful discussions and for her continued support during the writing stage.

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Received 5 April 2006.

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