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PUSHING THE LIMIT: HOW FAR CAN WE GO?
(GENERALIZED LIMITS AND LIMIT EXTREMA IN TOPOLOGY)

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Abstract. The traditional epsilon-delta definition of a limit is for *functions* in *metric* spaces. This paper defines limits for *relations* in *topological* spaces, and agrees with the usual definition in the original cases. The advantage is that less information is needed and more interesting cases are covered, as the functional dependence on a "variable" is unnecessary for evaluating the limits of many expressions (for example, the limit of a Riemann sum as the norm of its partition approaches zero: the relation between the norm and the sum is not functional, yet the limit is naturally defined). The general definition is in some ways simpler, providing enlightening perspectives on traditional uses of limits, and a context for the generalization of limit extrema (limsup and liminf) in unordered spaces via a concept of "limit closure."

0. **Introduction.** Section 1 of this paper installs a generalized theory of limits, not only for the purposes of Section 2, but also to clear up a great deal of complications (and hand-waving) in introductory analysis and calculus. After limits and limit methods are introduced in first-year calculus, there may be a tendency in more advanced courses to only reintroduce the definition, and not the methods. For this reason, some of the methods may go unproven or poorly generalized.

As well, the epsilon-delta definition of a limit is very method-oriented, in the sense that it is stated so as to be suggestive of a standard pattern for proofs (i.e., pick an epsilon, and find a delta). This is unfortunate, because generalization can lead to a much simpler understanding (although the usual methods for *dealing* with limits remain useful).

That is why Section 1 of this paper can be beneficial to any undergraduate student interested in a clearer understanding of analysis. Indeed, these ideas were originally compiled purely to enhance my own understanding. Even the context created here is helpful in considereing other concepts which this paper does not. There is often a temptation (and a good reason) to write expressions and make claims like

$$\lim_{x^2\to 0} \sin(x) = 0,\ \lim_{\substack{(x,y)\to 0\\ x=2y}} \left(\frac{y}{x}\right) = \frac{1}{2},\ \text{or} \lim_{x\to\infty} f\left(\frac{1}{x}\right) = \lim_{x\to 0^+} f(x).$$

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Rather than giving modified definitions and justifications for each of these ideas, it turns out that we can employ a unified treatment for all of them. As well, limits at infinity no longer require a seperate definition. Again, all of this is explored in the first section.

Section 2 is aimed at generalizing results about limit extrema in $\mathbb{R}$. Those unfamiliar with $\limsup$ and $\liminf$ should recall the ubiquitous "squeeze" theorem. The "limit supremum" of a function at a point c can be defined as the limit of the supremum of the function as we restrict our attention (i.e., the domain) to an open interval closer and closer to c. The "limit infimum" is the analogous infimum.

Intuitively, if $\limsup\limits_{x\to c} f(x) = \liminf\limits_{x\to c} f(x) = L$, then $f(x)$ is "squeezed between" its extrema, and $\lim\limits_{x\to c} f(x)$ is forced to "exist" at L. Conversely, if $\lim\limits_{x\to c} f(x) = L$, then the extrema of f as $x \to c$ are "pulled inward" and thus $\limsup\limits_{x\to c} f(x) = \liminf\limits_{x\to c} f(x) = L$.

Unfortunately, the definitions of $\limsup$ and $\liminf$ rely on the ordering of the range $\mathbb{R}$ (as we take suprema and infima), and are thus meaningless in unordered spaces (like $\mathbb{R}^n$, for example). However, since our intuitive grasp of the above mentioned results does not seem to depend on the ordering of the reals, (only on "squeezing" and "pulling"), it seems feasible that these results could have more general analogues, and in fact they do.

This concludes the introduction. Before proceeding to Section 1, for convenience, we will adopt the following conventions, listed here in bullet form for ease of reference from other parts of the paper:

- $\mathcal{X}$ and $\mathcal{Y}$ will always denote (topological) spaces.
- A, B, will always be subsets of $\mathcal{X}$.
- f is a function, and R is a relation.
- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ denotes the "extended reals."
- For a set T, $R[T] := \{u \mid \exists\ t; T, (t, u); R\}$ (and likewise for $f[T]$).
- For a point ρ, we will sometimes use ρ and $\{\rho\}$ interchangeably, e.g. by writing $R[\rho]$ instead of $R[\{\rho\}]$. In this way, $R[\ ]$ can be viewed partially as a multi-valued (i.e. set-valued) function (from $\mathcal{X}$ to the power set of $\mathcal{Y}$).
- *Neighborhood* will always mean *open neighborhood*.
- For a point, ρ, and a space S (not necessarily containing ρ, but at least contained in some unspecified larger space $T \ni \rho$), the expression $N_{\rho;S}$ will be a variable ranging over the set of all (open) neighbourhoods of ρ intersected with S. So, a clause like $\forall N_{\rho;S}$ is read "for all neighbourhoods of ρ in S" or "for all S-neighbourhoods of ρ". Note that for $S \subseteq T$, by definition, $N_{\rho;S} = N_{\rho;T} \cap S$. Also, if the space S is clear from context, we may write simply N_ρ.
- A raised index will be used to indicate a *particular* neigbhourhood of ρ (usually selected for some purpose), e.g. N_ρ^1.

1. **Generalized Limits.** To begin, we must first acknowledge that when considering a limit of a function (or a relation, as we shall see later) defined on some domain, $\mathcal{X}$, the point at which the limit is taken, c, should be "approachable" in the sense that there should be points arbitrarily close to c for which the function

(or relation) is defined. It would not make much sense (in the real case) to consider the limit of $\log x$ as x approaches -3.

Consequently, for a space $\mathcal{S}$ and a relation R, the set of points *approachable by R through $\mathcal{S}$* is defined as

$$\boxed{\begin{aligned} \mathrm{Ap}_{\mathcal{S}}R &:= (\mathcal{S} \cap \text{ Dmn } R)' \\ &= \{c : \forall N_{c;\mathcal{S}},\ \text{Dmn } R \cap N_{c;\mathcal{S}} \setminus c \neq \emptyset\} \\ &= \{c : \forall N_{c;\mathcal{S}}, R[N_{c;\mathcal{S}} \setminus c] \neq \emptyset\} \end{aligned}}$$

The points, c, are taken from $\mathcal{S}$ or an ambient space $T \supseteq \mathcal{S}$, which may go unmentioned (for $c \in \mathrm{Ap}_{\mathcal{S}}R$, it is required only that $N_{c;\mathcal{S}}$ be appropriately defined).

We now proceed to the definition of the limit. To give set-theoretic precision to their "existence", limits are defined as set-valued operators. For topological spaces $\mathcal{X}$ and $\mathcal{Y}$, a relation R, $R[\mathcal{X}] \subseteq \mathcal{Y}$ (the relation takes $\mathcal{X}$ into $\mathcal{Y}$) and a point $c \in \mathrm{Ap}_{\mathcal{X}}R$, all these assumptions are made implicitly to write

$$\boxed{{}^{\mathcal{Y}}_{\mathcal{X}}\lim_{x\to c} R[x] := \{L \in \mathcal{Y} \mid \forall N_{L;\mathcal{Y}}\ \exists\ N_{c;\mathcal{X}} : R[N_{c;\mathcal{X}} \setminus c] \subseteq N_{L;\mathcal{Y}}\}}$$

which is read "the limit through $\mathcal{X}$ to $\mathcal{Y}$ of $R[x]$ as x approaches c". When the variable x is unnecessary, we write ${}^{\mathcal{Y}}_{\mathcal{X}}\lim_{c} R$, and when $\mathcal{X}$ and $\mathcal{Y}$ are implicit, simply $\lim_{c} R$ is sufficient.[1]

The assumption that $c \in \mathrm{Ap}_{\mathcal{X}}R$ warrants special attention. If the definition were made without this assumption, it might be that $R[N^1_{c;\mathcal{X}} \setminus c] = \emptyset$ for some $N^1_{c;\mathcal{X}}$, in which case *every* point in $\mathcal{Y}$ would be in the limit, since for each $L \in \mathcal{Y}$, and any $N_{L;\mathcal{Y}}$, $R[N^1_{c;\mathcal{X}} \setminus c] \subseteq N_L$, so L is in the limit. This is an uninteresting and troublesome case to consider, so it is ruled out here at the outset (it would be rather annoying to always have to say "unless the limit is all of $\mathcal{Y}$"). This also explains the choice of definition for $\mathrm{Ap}_{\mathcal{X}}R$ instead of simply using Dmn R.

In the case where $\mathcal{X}$ and $\mathcal{Y}$ are metric spaces (such as $\mathbb{R}^n$), the above definition corresponds with traditional limits in the sense that

$$\begin{aligned} & \forall N_L\ \exists\ N_L : f[N_c \setminus c] \subseteq N_L \\ \iff & \forall \epsilon > 0\ \exists\ \delta > 0 : f[B_\delta(c) \setminus c] \subseteq B_\epsilon(L) \\ \iff & \forall \epsilon > 0\ \exists\ \delta > 0 : 0 < |x - c| < \delta \Rightarrow |f(x) - L| < \epsilon \end{aligned}$$

where $B_r(\rho)$ denotes the open ball of radius r about ρ. Thus the generalized limit will be a singleton set $\{L\}$, when the limit "exists", and $\emptyset$ when the limit "does not exist". In fact, the prized uniqueness of limits (when they "exist") is in general contingent only on $\mathcal{Y}$ being Hausdorff.[2] Intuitively, we cannot "force" the values of R (as a multi-valued function) to be in two places at once:

Theorem (1.1)**.** *For $\mathcal{Y}$ Hausdorff,* $|\lim_{c} R| \leq 1$.

Proof: Suppose $L, K \in \lim_{c} R$, $L \neq K$. Take N^1_L, N^2_K such that

$$N^1_L \cap N^2_K = \emptyset.$$

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[1] In this article, all limits are ${}^{\mathcal{Y}}_{\mathcal{X}}$ (that is, from $\mathcal{X}$ to $\mathcal{Y}$) unless otherwise indicated, and all points and neighbourhoods are in the appropriate spaces when no confusion results.

[2] A space is called Hausdorff iff $\forall x \neq y,\ \exists N_x, N_y\ :\ N_x \cap N_y = \emptyset$.

We have

$$\exists N_c^1 : R[N_c^1 \setminus c] \subseteq N_L^1 \quad \text{and} \quad \exists N_c^2 : R[N_c^2 \setminus c] \subseteq N_K^2.$$

But from general set theory, $R[S \cap T] \subseteq R[S] \cap R[T]$, hence

$$R[(N_c^1 \cap N_c^2) \setminus c] \subseteq R[N_c^1 \setminus c] \cap [N_c^2 \setminus c] = \emptyset,$$

contradicting that $c \in \mathrm{Ap}_{\mathcal{X}} R$. The result follows. $\square$

Following the common algebraic convention of using an element and its singleton interchangeably, we may write $\lim\limits_{c} = L$ instead of $\{L\}$ when no confusion results. Thus all traditional uses of limits can be seen as instances of the generalization.

Having these definitions firmly in place, we are never far from home when writing expressions like

$$\lim_{x^2 \to 0} \sin(x) = 0$$

since $\{(x^2, \sin x) | x \in \mathbb{R}\}$ *is* a relation (though not a function), or

$$\lim_{\substack{(x,y)\to 0 \\ x=2y}} \left(\frac{x}{y}\right) = 2$$

since $x = 2y$ defines a subset $\mathcal{X} \subseteq \mathbb{R}^2$, and furthermore we have the relation $R = \{\left((x,y), \frac{x}{y}\right) : 0 \neq x, y \in \mathbb{R}\}$ with $0 \in \mathrm{Ap}_{\mathcal{X}} R$ (these notations will not be used as conventions in this article, except in some recurring examples).

As well, $\overline{\mathbb{R}}$ is topologically identical (homeomorphic) to the interval $[-1, 1]$ via the map $x \mapsto \dfrac{x}{1-x^2}$, $-1 \mapsto -\infty$, $1 \mapsto \infty$. Thus, limits at $\pm\infty$ require no seperate treatment, since $\pm\infty$ have neighbourhoods just as "finite" reals do (specifically, a N_∞ is an open set containing some ray $(a, \infty]$, and a $N_{-\infty}$ is an open set containing some ray $[-\infty, a)$).

The following formally justifies a "change of variables" in a limit.

Theorem (1.2)**.** *If $h : \mathcal{X}_1 \to \mathcal{X}_2$ is a homeomorphism, then*

$$\,^{\mathcal{Y}}_{\mathcal{X}_1}\lim_{x \to c} R[x] = \,^{\mathcal{Y}}_{\mathcal{X}_2}\lim_{x \to h(c)} R[h^{-1}(x)], \quad \textit{i.e.},$$

$$\,^{\mathcal{Y}}_{\mathcal{X}_1}\lim_{c} R = \,^{\mathcal{Y}}_{\mathcal{X}_2}\lim_{h(c)} R \circ h^{-1}.$$

Proof: Suppose $L \in \,^{\mathcal{Y}}_{\mathcal{X}_1}\lim\limits_{c} R$. For any N_L we have some N_c and hence $h[N_c] = N_{h(c)}$ such that

$$R \circ h^{-1}[N_{h(c)}] = R \circ h^{-1}[h[N_c]] = R[N_c] \subseteq N_L,$$

so $L \in \,^{\mathcal{Y}}_{\mathcal{X}_2}\lim\limits_{h(c)} R \circ h^{-1}$, and by symmetry the result follows. $\square$

Hence, by the above, a claim like $\lim\limits_{x\to\infty} f\left(\dfrac{1}{x}\right) = \lim\limits_{x \to 0^+} f(x)$ is justified, since $h : [0, \infty] \to [0, \infty]$ by $x \mapsto \frac{1}{x}$, $0 \mapsto \infty$, $\infty \mapsto 0$, is a homeomorphism.

The next result gives a corollary which generalizes the technique of showing when limits do not exist by showing conflicting subspace limits:

Theorem (1.3)**.** *If $A \subseteq B$, then ${}^{\mathcal{Y}}_{A}\lim_{c} R \supseteq {}^{\mathcal{Y}}_{B}\lim_{c} R$ (in particular, when $B = \mathcal{X}$, ${}^{\mathcal{Y}}_{A}\lim_{c} R \supseteq \lim_{c} R$).*

> **Proof:** Suppose $L \in {}^{\mathcal{Y}}_{B}\lim_{c} R$. For any N_L, we have some N_c which gives $R[A \cap N_c \setminus c] \subseteq R[B \cap N_c \setminus c] \subseteq N_L$. Thus $L \in {}^{\mathcal{Y}}_{A}\lim_{c} R$, hence the result. □

Corollary (1.4)**.** *If $A, B \subseteq \mathcal{X}$ and ${}^{\mathcal{Y}}_{A}\lim_{c} R \cap {}^{\mathcal{Y}}_{B}\lim_{c} R = \emptyset$, then $\lim_{c} R = \emptyset$.*

> **Proof:** We have ${}^{\mathcal{Y}}_{A}\lim_{c} R \supseteq \lim_{c} R \subseteq {}^{\mathcal{Y}}_{B}\lim_{c} R$, hence
> $$\lim_{c} R \subseteq \left({}^{\mathcal{Y}}_{A}\lim_{c} R \right) \cap \left({}^{\mathcal{Y}}_{B}\lim_{c} R \right) = \emptyset,$$
> so $\lim_{c} R = \emptyset$. □

For instance, since

$$\lim_{\substack{(x,y)\to 0 \\ x=2y}} \frac{x}{y} = 2 \neq 1 = \lim_{\substack{(x,y)\to 0 \\ x=y}} \frac{x}{y},$$

we conclude that $\lim_{(x,y)\to 0} \frac{x}{y} = \emptyset$, which is an accurate and consise way of writing "the limit does not exist".

2. **Limit Closure — Generalizing Limit Extrema.** Although lim sup and lim inf are usually defined for functions from $\mathbb{R}$ to $\overline{\mathbb{R}}$, these operators work equally well on relations, with the appropriate definition. We choose one that makes few assumptions:

For $c \in \mathrm{Ap}_{\mathcal{X}} R$, and $\mathcal{Y} = \overline{\mathbb{R}}$,

$$\limsup_{c} R := \inf_{N_c}(\sup(R[N_c \setminus c])),$$

and similarly for lim inf.

We record as a theorem the following result, a minor modification of the usual result for functions:

Theorem (2.1)**.** $\limsup_{c} R = \liminf_{c} R = L \iff \lim_{c} R = L.$

Intuitively, the limit extrema "squeeze" the values of R to a single point. To reproduce this phenomenon, we define the "limit closure through $\mathcal{X}$ to $\mathcal{Y}$ of $R[x]$ as $x \to c$" as follows (note the capital L):

$$\boxed{{}^{\mathcal{Y}}_{\mathcal{X}}\operatorname*{Lim}_{x\to c} R[x] := \bigcap_{N_{c;\mathcal{X}}} \overline{R[N_{c;\mathcal{X}} \setminus c]},}$$

favouring the notations ${}^{\mathcal{Y}}_{\mathcal{X}}\operatorname*{Lim}_{c} R$ and $\operatorname*{Lim}_{c} R$ when possible. (Here, all limit closures will be ${}^{\mathcal{Y}}_{\mathcal{X}}$ unless otherwise specified).

By relating limit closures to limit extrema, this theorem affirms the motivation for the chosen definition:

Theorem (2.2)**.** *For* $\mathcal{Y} = \overline{\mathbb{R}}$*,* $\limsup\limits_{c} R = \max \operatorname*{Lim}\limits_{c} R$*, and likewise* $\liminf\limits_{c} R = \min \operatorname*{Lim}\limits_{c} R$*.*

Proof: Since $\operatorname*{Lim}\limits_{c} R$ is the intersection of a collection of closed sets, no finite number of which are disjoint, we have $\operatorname*{Lim}\limits_{c} R \neq \emptyset$ (this result will be demonstrated in more detail and generality in Theorem 2.6). Now, since $\operatorname*{Lim}\limits_{c} R$ is closed, the results are straightforward:

$$\begin{aligned} \max \operatorname*{Lim}_{c} R &= \limsup_{c} R = \sup \bigcap_{N_c} \overline{R[N_c \setminus c]} \\ &= \inf_{N_c} (\sup \overline{R[N_c \setminus c]}) = \inf_{N_c} (\sup R[N_c \setminus c]) = \limsup_{c} R \end{aligned}$$

and similarly for the second result. □

Hence, reconsidering Theorem 2.1, we have the equivalent

Corollary (2.3)**.** For $\mathcal{Y} = \overline{\mathbb{R}}$, $\lim\limits_{c} R = \{L\} \Longleftrightarrow \operatorname*{Lim}\limits_{c} R = \{L\}$.

We will now endeavor to examine the relationship between limits and limit closures in a general topological setting, beginning with the following useful

Lemma (2.4)**.** $\operatorname*{Lim}\limits_{c} R = \{L \in \mathcal{Y} : \forall N_c \forall N_L, N_L \cap R[N_c \setminus c] \neq \emptyset\}$.

Proof:

$$\begin{aligned} L \in \operatorname*{Lim}_{c} R &\Longleftrightarrow L \in \bigcap_{N_c} \overline{R[N_c \setminus c]} \\ &\Longleftrightarrow \forall N_c, L \in \overline{R[N_c \setminus c]} \\ &\Longleftrightarrow \forall N_c \forall N_L, N_L \cap R[N_c \setminus c] \neq \emptyset, \end{aligned}$$

hence the result. □

Intuitively, a limit $\lim\limits_{c} R$ returns values that the relation "gets close to" and "stays close to" as $x \to c$, whereas from the Lemma it is now clear that the limit closure, $\operatorname*{Lim}\limits_{c} R$, returns values that the relation "gets close to" but need not "stay close to". For example, as $x \to 0$, the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = (1 - x^2)\sin(x^{-1})$ can be said to "get close to 1" abitrarily often, but never "stays close to 1", and in fact $\operatorname*{Lim}\limits_{c} f = [-1, 1]$ in this case. Since "getting and staying close to c" is a stronger property than only "getting close to c," it is now unsurprising that we have in general:

Theorem (2.5)**.** *For any space* $\mathcal{Y}$*,* $\lim\limits_{c} R \subseteq \operatorname*{Lim}\limits_{c} R$*.*

Proof: Suppose $L \in \lim\limits_{c} R$. For each N_L, choose N_c^1 so that

$$R[N_c^1 \setminus c] \subseteq N_L,$$

and hence $\forall N_L$ we have

$$R[N_c \setminus c] \supseteq R[N_c \cap N_c^1 \setminus c] \subseteq R[N_c^1 \setminus c] \subseteq N_L,$$

thus $N_L \cap R[N_c \setminus c] \neq \emptyset$ (note $R[N_c \cap N_c^1 \setminus c] \neq \emptyset$ since $c \in \mathrm{Ap}_{\mathcal{X}} R$) so $L \in \operatorname*{Lim}_c R$ by Lemma 2.4, and the result follows. □

In Section 1, the condition that $\mathcal{Y}$ is Hausdorff was shown to place an upper bound on the size of $\lim_c R$. There is, analogously, a condition that offers a *lower* bound on the size of $\operatorname*{Lim}_c R$:

Theorem (2.6)**.** *For $\mathcal{Y}$ compact, $|\operatorname*{Lim}_c R| \geq 1$.*

Proof: Suppose $\operatorname*{Lim}_c R = \emptyset$. Then, taking complements (indicated by raised $\sim$'s to avoid confusion with c) of the sets $\overline{R[N_c \setminus c]}$, we find that

$$\bigcup_{N_c} \overline{R[N_c \setminus c]}^{\sim} = \left(\bigcap_{N_c} \overline{R[N_c \setminus c]} \right)^{\sim} = (\operatorname*{Lim}_c R)^{\sim} = \emptyset^{\sim} = \mathcal{Y}$$

gives an open cover of $\mathcal{Y}$, and so we may choose a finite subcover given by a finite number of N_c's, say $N_c^1, N_c^2, \ldots, N_c^n$. That is,

$$\bigcup_{k=1}^{n} \overline{R[N_c^k \setminus c]}^{\sim} = \mathcal{Y}, \ \therefore \bigcap_{k=1}^{n} \overline{R[N_c^k \setminus c]} = \emptyset.$$

But $\bigcap_{k=1}^{n} N_c^k$ is a neighbourhood of c, so, recalling that $c \in \mathrm{Ap}_{\mathcal{X}} R$, we now have

$$\emptyset = \bigcap_{k=1}^{n} \overline{R[N_c^k \setminus c]} \supseteq \bigcap_{k=1}^{n} R[N_c^k \setminus c] \supseteq R\left[\bigcap_{k=1}^{n} N_c^k \setminus c \right] \neq \emptyset,$$

a contradiction, and the result follows. □

To illustrate the necessity of the condition that $\mathcal{Y}$ be compact, consider $f : \mathbb{R} \to \mathbb{R}$ by $x \mapsto \frac{1}{x}$, $0 \mapsto 0$. Then it is not hard to see that $\operatorname*{Lim}_{x \to 0} f(x) = \emptyset$.

For $\mathcal{Y}$ compact, it is naturally interesting to consider the minimal case where $\operatorname*{Lim}_c R = \{L\}$. Since $\lim_c R \subseteq \operatorname*{Lim}_c R$ in general, either $\lim_c R = \emptyset$ or $\lim_c R = \{L\}$. It turns out that the second case is the only possibility. Intuitively, since the values of R never "get close" to anything but L, they are forced to "stay close" to L:

Theorem (2.7)**.** *For $\mathcal{Y}$ compact, $\operatorname*{Lim}_c R = \{L\} \Rightarrow \lim_c R = \{L\}$.*

Proof: Consider an arbitrary N_L. $(N_L)^{\sim}$ is closed and therefore compact. Now, for any $K \in \mathcal{Y}$ with $K \neq L$, we have $K \notin \operatorname*{Lim}_c R$, which by Lemma 2.4, translates as

$$\forall K \neq L \ \exists N_c^K \ \exists N_K, \ N_K \cap R[N_c^K \setminus c] = \emptyset, \text{ i.e. } R[N_c^K \setminus c] \subseteq (N_K)^{\sim}.$$

We now have the open covering $(N_L)^{\sim} \subseteq \bigcup_{K \notin N_L} N_K$, which then has a finite subcovering given by some $K_1, K_2, \ldots, K_n$, i.e.

$(N_L)^{\sim} \subseteq \bigcup_{i=1}^{n} N_{K_i}$, and thus $N_L \supseteq \bigcap_{i=1}^{n} (N_{K_i})^{\sim}$.

Next, taking $N_c = \bigcap_{i=1}^{n} N_c^{K_i}$, we have

$$R[N_c \setminus c] = R\left[\bigcap_{i=1}^{n} N_c^{K_i} \setminus c\right] \subseteq \bigcap_{i=1}^{n} R\left[N_c^{K_i} \setminus c\right] \subseteq \bigcap_{i=1}^{n} (N_{K_i})^{\sim} \subseteq N_L.$$

Thus, for each N_L there is an appropriate N_c given by the above construction and it follows that $L \in \lim\limits_c R$. But $\lim\limits_c R \subseteq \operatorname*{Lim}\limits_c R = \{L\}$, so we must have $\lim\limits_c R = \{L\}$. □

The implication $\lim\limits_c R = \{L\} \Longrightarrow \operatorname*{Lim}\limits_c R = \{L\}$, along with its converse, form Corollary 2.3 when $\mathcal{Y} = \overline{\mathbb{R}}$. Now that the compactness of $\mathcal{Y}$ has been shown sufficient for the forward implication, we seek conditions under which the converse holds. In this case, the proof turns out to be much less involved:

Theorem (2.8)**.** *For $\mathcal{Y}$ Hausdorff, $\lim\limits_c R \neq \emptyset \Longrightarrow \operatorname*{Lim}\limits_c R \subseteq \lim\limits_c R$, i.e., by the uniqueness of limits in Hausdorff spaces and the fact that $\lim\limits_c R \subseteq \operatorname*{Lim}\limits_c R$,*

$$\lim_c R = \{L\} \Longrightarrow \operatorname*{Lim}_c R = \{L\}$$

Proof: Suppose $K \notin \lim\limits_c R = \{L\}$, i.e. $K \neq L$. Take $N_K^1 \cap N_L^1 = \emptyset$ by the Hausdorff property. Take $N_c^1 : R[N_c^1 \setminus c] \subseteq N_L^1$. Then, clearly

$$R[N_c^1 \setminus c] \cap N_k^1 = \emptyset$$

hence by Lemma 2.4, we have $K \notin \operatorname*{Lim}\limits_c R$, and then it follows that $\operatorname*{Lim}\limits_c R \subseteq \lim\limits_c R$ (i.e. $\operatorname*{Lim}\limits_c R = \{L\}$). □

Now, for the case where $\mathcal{Y}$ is both compact *and* Hausdorff, we have both directions of implication, and so Theorem 2.1, in the equivalent form of Corollary 2.3, can be generalized for any compact Hausdorff space, $\mathcal{Y}$. The results are summarized in the convenient collection below.

Theorem (2.9)**.** *Properties of limits in a compact Hausdorff space, $\mathcal{Y}$:*

a) $\lim\limits_c R = \{L\} \Longleftrightarrow \operatorname*{Lim}\limits_c R = \{L\}$,
b) $|\lim\limits_c R| \leq 1$ *and* $|\operatorname*{Lim}\limits_c R| \geq 1$,
c) $|\lim\limits_c R| = 1 \iff |\operatorname*{Lim}\limits_c R| = 1 \iff \operatorname*{Lim}\limits_c R = \lim\limits_c R$.

Proof:
a) Follows from Theorems 2.7 and 2.8.
b) Restatement of Theorems 1.1 and 2.6.
c) First,

$$\begin{aligned} |\lim_c R| = 1 \quad &\iff \quad \lim_c R = \{L\} \quad (L \text{ chosen}) \\ &\iff \quad \operatorname*{Lim}_c R = \{L\} \\ &\iff \quad |\operatorname*{Lim}_c R| = 1, \end{aligned}$$

and so, also, any of these implies $\lim\limits_c R = \operatorname*{Lim}\limits_c R$. Second, when $\operatorname*{Lim}\limits_c R = \lim\limits_c R$, by (b) we must have $|\operatorname*{Lim}\limits_c R| = |\lim\limits_c R| = 1$, completing the proof. □

3. **Concluding Remarks.** Since Corollary 2.3 is equivalent to Theorem 2.1, the final results of Section 2 (in particular, Theorem 2.9a) can be seen as a generalization of the usual results about limit extrema. As for generalizing $\limsup_c R$ and $\liminf_c R$ themselves, they together form the boundary of $\operatorname*{Lim}_c R$ in the real case, and so $\mathrm{bd}(\operatorname*{Lim}_c R)$ is an appropriate contender.

It is worth noting that in the case where $\mathcal{Y}$ is a compact Hausdorff space, all information about the values of the lim operator can be deduced from those of the Lim operator, whereas the reverse is not true: the limit closure includes information about the function which the usual limit does not. This is an interesting advantage of the limit closure concept from a theoretical standpoint, which of course echoes the advantages of working with limit extrema in the real-valued case.

Furthermore, even when $\mathcal{Y} = \mathbb{R}$, the limit extrema of two functions f and g may coincide at a point while the limit closures at the point are different. For example, consider $f(x) = \sin(x^{-1})$ and $g(x) = [\sin(x^{-1})]$ (where $[\cdot]$ is the nearest integer operator): $\operatorname*{Lim}_0 f = [-1, 1]$ but $\operatorname*{Lim}_0 g = \{-1, 0, 1\}$. Hence the limit closure of a function at a point is even more descriptive than the limit extrema at the point.

Returning to the broader setting of Section 1, now that we have a topological meaning for the statement "R approaches L at c," from a logical standpoint it would be convenient to define what it means for a "variable," x, to approach a constant, c. Notice that, hitherto, the expression "$x \to c$" has not been given meaning as a stand-alone statement. If it were, then transitive reasoning like "$x \to c$, therefore $f(x) \to L$, therefore $g(f(x)) \to M$" might be a valid and natural pattern of thought. To a limited extent, a "variable" in this way can be thought of as a sequence, and the concepts of "nets" and "filters" are somewhat more effective in developing this idea. The reader is referred to [1] and [2], however I suspect that a more general (and more elegant) approach is possible.

References / Recommended Reading

[1] Bourbaki, N., *Elements of Mathematics: General Topology, Chapters 1-4*, Springer-Verlag (1989), pp. 57-74.

This section of the text is recommended as an introduction to filters in topological spaces. Also, this is a very comprehensive reference book for topology.

[2] Kelly, J.L., *General Topology*, Van Nostrand (1955), pp. 62-83.

Nets, which are a generalization of sequences, are covered in this chapter of the text, entitled "Moore-Smith Convergence."

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