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HOW DO MATHEMATICS AND POKER MIX?

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I am asked frequently how I view the role of mathematics in poker. I see three distinct ways in which mathematics relates to poker. The first is through the mathematics one actually uses while playing the game. The second is through the mathematics that changes or reinforces how one thinks about the game. The third is via the wide range of mathematical questions that arise from poker. I shall discuss each of these in turn. Also, I am assuming that the reader has some familiarity with the standard poker games and terminology. Anyone who wishes to brush up on poker information should go to the Canadian Poker Player Magazine website (http://www.canadianpokerplayer.com) and follow some of the links.

Issues involving pot odds form the broadest application of mathematics during actual play. The idea is straightforward. When a player is facing a bet of \$x and the pot has \$y in it, we say the pot is offering her pot odds of y-to-x. If she estimates that the odds of winning the pot are better than y-to-x, then calling the bet of \$x is going to be profitable in the long run. Here is a typical example. A player has A-J of hearts and the board has 2-6-8-Q of which two are hearts. If the pot has \$100 and she is facing a bet of \$10, then the pot is offering her 10-to-1 pot odds. The odds against catching a heart on the river are about 4-to-1. The odds against catching a heart that doesn't pair the board are about 5.5-to-1.

She can expect to catch a heart about one in every five times she is in this situation. Thus, she will make a profit of \$100 about one-fifth of the time and lose her \$10 about four-fifths of the time. This means she is averaging about a \$12 profit for every bet. This assumes she wins if a heart comes and she loses if it doesn't. This is a simplification, but in any case, it is a very profitable situation and she should call the bet.

The preceding example is clearcut but that is not always the case. Sometimes it can be tricky to figure out the odds against winning. You must estimate the strengths of your opponents' hands. You also must estimate what kind of future action you might get should you hit a card you need. Making these estimates is part of the mathematical approach.

Players also can manipulate the pot odds being offered to other players. For example, if there are two players left and the first to act suspects her opponent is on a flush draw, then a raise roughly equal to the size of the pot means that her opponent is getting pot odds of only about 2-to-1. Thus, she has placed her opponent in the position of not having the proper odds to call should her opponent be on some kind of draw. Of course, this applies mostly to pot limit and no-limit games. In limit games there are many instances when the pot is big enough that essentially all the players have the correct pot odds to call.

Mathematics also is used in certain contexts to introduce randomization to a player's arsenal. In many low limit games, your opponents may not be paying all that much attention to patterns in your betting. However, if you are up against one or more astute players, you may want to vary your betting to some degree. For example, if you are in early position with A-K offsuit and any players acting before you have folded, you might decide to make an 85-15 random mix of raising and limping. That is, 85% of the time you will raise and 15% of the time you will limp. One way you can essentially achieve this is as follows. There are 12 different ways of holding A-K offsuit. Whenever both cards are red, you will limp. In all other cases you will raise. This means the probability of limping is 1/6 and the probability of raising is 5/6. This method gives no hint to the other players what you are doing since your randomization scheme does not depend on anything external to your hand.

Let's leave the mathematics one employs while playing the game and turn to the mathematics that changes or reinforces how we think about the game. This kind of mathematics normally is done away from the heat of battle because the mathematics normally is beyond the kind of thing one can do in one's head. Sometimes the results are surprising or counterintuitive as well.

The next table serves as an example of this kind of mathematics. Let me describe the entries in the table. The column headed "pair" is the rank of a pair some player holds as her hand. The column headed "one rank" is the probability that there is exactly one bigger rank y with one or two of her nine opponents holding a pair of rank y. Similarly, the column headed "two ranks" is the probability that there are exactly two ranks y and z with one or two opponents holding a pair of rank y, and one or two opponents holding a pair of rank z.

pair	one rank	two ranks	three ranks
K-K	.0439	-	-
Q-Q	.08412	.001863	-
J-J	.12161	.005435	.0000768
10-10	.15519	.01044	.0002844
9-9	.1857	.01662	.0007109
8-8	.2132	.02397	.001367
7-7	.2380	.03218	.0023
6-6	.2603	.04114	.003538
5-5	.2801	.05071	.0051
4-4	.2977	.06076	.007
3-3	.3133	.07118	.009251
2-2	.3269	.08186	.01186

Contemplating the information contained in the preceding table should cause any hold'em player to think again about the value of certain pairs. For example, a player with a pair of deuces is going to be up against a single larger rank, with one or two opponents holding a pair of that rank, about one-third of the time. About one out of 12 times the pair of deuces will be facing two larger ranks with pairs of these ranks dealt. On the other hand, a player with pocket queens will have the top pair dealt about 11 out of 12 times.

When I derived the numbers in the preceding table, I did not filter out the deals in which two players are dealt pairs of the same rank. This is a rare occurrence so that most of the time a player is facing just a single pair of a given rank.

There are other examples of mathematics providing poker insight, but let's now turn to mathematical problems arising in a poker context. These are problems that have little effect on playing poker, but frequently are of mathematical interest. Here are a few examples.

I was once asked for the probability that a player will win a royal flush bonus in hold'em given that both of the player's hole cards must play. This is an easy counting problem so let's solve it. The total number of boards for hold'em is $\binom{52}{5} = 2,598,960$. To determine how many allow the royal flush bonus, we observe that there are four choices for the suit and 10 choices for three out of the five royal cards in that suit. The remaining two cards may be any two of the other 47 cards. Thus, there are $4 \cdot 10\binom{47}{2} = 43,240$ boards allowing a royal flush bonus. Dividing by 2,598,960 yields a probability of 1,081/64,974 that the board allows a royal flush bonus. Now a given player has $\binom{47}{2} = 1,081$ possible two-card hands. It follows that a given player has a probability of 1/1,081 that she holds the two magic cards giving her a royal flush. Multiplying yields a probability of 1/64,974 that you will win a royal flush bonus in any hand of hold'em. In short, don't count on this bonus to pay your phone bill.

Knowing this answer is of no strategic importance to a player, though it does satsify some people's curiosity. On the other hand, this question is of importance to managers who are trying to determine the cost of offering a royal flush bonus.

Another curiosity driven question deals with the chances of a player being dealt back-to-back pocket aces. When someone asks "What are the chances of being dealt back-to-back pocket aces," we have a perfect illustration of a question that cannot be answered as posed. Why do I say this? The problem with the question as asked is that it is subject to three interpretations (perhaps more for the imaginative reader) and there is insufficient information to decide which interpretation is intended.

Here is one interpretation. The player has just been dealt A-A so what is the probability she is dealt A-A on the next hand? This is easy to answer and is $1/221$. Another interpretation is to ask what is the probability she will be dealt A-A on her next two hands. This also is easy to answer and is $(1/221)^2$.

The most interesting interpretation is to ask what is the probability a player is dealt back-to-back A-A sometime during a session of n hands. This interpretation leads one into what are called *generating functions*. We get this by looking at the possible sequences of n hands a player may be dealt and develop a recurrence relation for those not containing back-to-back pocket aces. We do this as follows.

Because the total number of different hold'em hands a player may be dealt is $\binom{52}{2}) = 1,326$, the total number of sequences of n possible hands is 1,326 raised to the n-th power. Now we need to determine how many of those sequences do not contain back-to-back A-A hands.

Let $c(n)$ denote the number of ways of dealing n successive hold'em hands to a fixed player so that the player does not receive back-to-back pocket aces. Let $c(n;b)$ be the number of ways of dealing n successive hold'em hands without back-to-back pocket aces so that the n-th hand is not pocket aces. Let $c(n;a)$ be the number of ways of dealing n successive hold'em hands without back-to-back pocket aces so that the n-th hand is pocket aces. It is easy to see that $c(n) = c(n;a) + c(n;b)$. Now if the n-th hand is pocket aces, the preceding hand must not be pocket aces or else we would have back-to-back pocket aces. Thus, $c(n; a) = 6c(n-1, b)$. On the other hand, if the n-th hand is not pocket aces, then the preceding hand can be anything. Thus, $c(n; b) = 1,320c(n-1)$.

Performing a little algebra leads to $c(n) = 1,320c(n-1) + 7,920c(n-2)$. We may solve this recurrence relation in exactly the same way we solve the recurrence relation for the Fibonacci sequence. Doing so leads to

$$\begin{aligned} c(n) \quad &= \quad (663 + \frac{669\sqrt{770}}{28})(660 + 24\sqrt{770})^{n-1} + \\ &\quad\;\; (663 - \frac{669\sqrt{770}}{28})(660 - 24\sqrt{770})^{n-1}. \end{aligned}$$

The preceding, in turn, allows us to give the exact probability for a player to be dealt back-to-back aces over a session of n hands. For example, if you play a session of 200 hold'em hands, the probability is about 1/250 that you will receive back-to-back pocket aces sometime during the session. You can see that it is not a common event.

For my last example, here is a question about Omaha. Most poker players know that there are 169 different hold'em hands, where we are ignoring suits. In fact, many players can tell you how to get that number. If you ask these same players how many Omaha hands there are, you will get a blank stare. The ad hoc method that works to get 169 hold'em hands is too messy to carry out for Omaha. On the other hand, mathematically counting hands ignoring suits really means that you are letting the group formed from all 24 possible permutations of the four suits act on the total number of 4-card hands (Omaha hands). What you need to do is count the number of orbits under the action of this particular permutation group. You apply a lemma named after Burnside and obtain 16,432 possible Omaha hands. To see the details for this, go to the Poker Computations directory at my website (http://www.math. sfu.ca/~alspach) and look at the file entitled "Enumerating Starting Poker Hands."

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