Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Winter 2012

Solution to Assignment #4
Series business at last!

1. Show that the alternating harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges *without* using the Alternating Series Test. *[5]*

Solution. We carefully regroup the series:

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} &= \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2k+1} - \frac{1}{2k+2} - \dots \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) + \cdots - \left(\frac{1}{2k+1} - \frac{1}{2k+2}\right) - \dots \\
&= \frac{1}{1\cdot 2} + \frac{1}{3\cdot 4} + \frac{1}{5\cdot 6} + \cdots + \frac{1}{(2k+1)(2k+2)} + \dots \\
&= \sum_{k=0}^{\infty} \frac{1}{(2k+1)(2k+2)} = \sum_{k=0}^{\infty} \frac{1}{4k^2+6k+2}
\end{aligned}$$

Since for $k \geq 1$ we have $\displaystyle\frac{1}{4k^2+6k+2} < \frac{1}{4k^2} < \frac{1}{k^2}$, the regrouped series converges by comparison with $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$, which it's easy to show converges (the p-Test or the Integral Test will do the job, for example). ■

2. Suppose a_n is a non-increasing sequence of positive terms such that $\displaystyle\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. Show that $\displaystyle\sum_{n=0}^{\infty} a_n$ also converges. *[5]*

Solution. Suppose a_n is a non-increasing sequence of positive terms such that $\displaystyle\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. We carefully regroup the original series:

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n &= a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \dots \\
&= a_0 + (a_1) + (a_2 + a_3) + (a_4 + a_5 + a_6 + a_7) + (a_8 + a_9 + \dots \\
&= a_0 + \sum_{k=0}^{\infty} \left(a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1}\right)
\end{aligned}$$

The original sequence is positive and non-increasing, *i.e.* $0 < a_m \leq a_k$ whenever $m > k$, so $0 < a_{2^k} + a_{2^k+1} + \cdots + a_{2^{k+1}-1} \leq a_{2^k} + a_{2^k} + \cdots + a_{2^k} = 2^k a_{2^k}$. Since $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges, it follows by the Comparison Test that $\sum_{n=0}^{\infty} a_n$ converges as well. ■

NOTE: Both of these can be done with the help of some (different!) rewriting trickery and the Comparison Test.