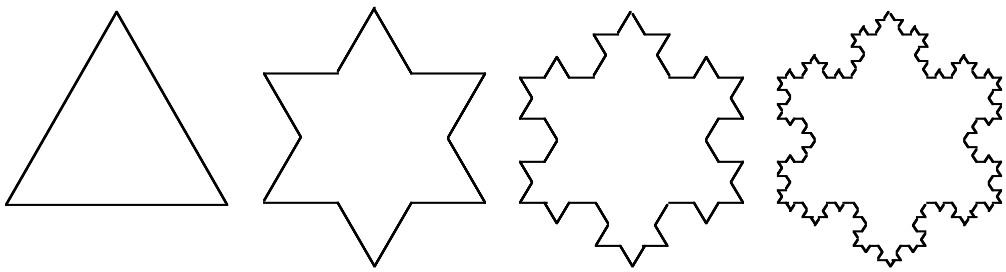
Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Winter 2012

Solutions to Assignment #1
A Seasonal Review

For this assignment you should probably crack open your old calculus textbook and skim through the material on sequences and series.

Suppose one has an equilateral triangle with sides of length 1. If one modifies each of the line segments composing the triangle by cutting out the middle third of the segment, and then inserting an outward-pointing "tooth," both of whose sides are as long as the removed third, one gets a six-pointed star. Suppose one repeats this process for each of the line segments making up the star, then to each of the line segments making up the resulting figure, and so on, as in the diagram:



Note that the lengths of the line segments at each stage are a third of the length of the segments at the preceding stage. For the sake of being definite, let's say we have the triangle at step 0 of the process, the six-pointed star at step 1 of the process, the next shape at step 2 of the process, and so on. The curve which is the limit of this process, if one takes infinitely many steps, is often called the *snowflake curve*. We will try to discover the length of this curve and the area of the region that it encloses below.

1. What is the length of the curve at step n of the process? *[2]*

SOLUTION. First, note that the *number* of line segments (disregarding their length for the moment) in the perimeter of the shape at step $n+1$ is 4 times the number of line segments in the perimeter of the shape at step n. Since we have 3 line segments at step 0, this means we have 12 at step 1, 48 at step 2, and so on. In general, we have $3 \cdot 4^n$ line segments at step n.

Second, note that the length of each individual line segment (disregarding how many there are for the moment) in the perimeter of the shape at step $n+1$ is $\frac{1}{3}$ times the length of each individual line segment in the perimeter of the shape at step n. Since the line segments at step 1 each have length 1, this means that each line segment at step 1 has length $\frac{1}{3}$, each at step 2 has length $\frac{1}{9}$, and so on. In general, each line segment in the perimeter of the shape at step n has length $\frac{1}{3^n}$.

It follows that the total length of the perimeter of the shape at step n is $3 \cdot 4^n \cdot \frac{1}{3^n} = 3 \cdot \frac{4^n}{3^n} = 3\left(\frac{4}{3}\right)^n$. ■

2. Find the length of the curve after all the infinitely many steps of the process. *[3]*

SOLUTION. The length of the curve after all the infinitely many steps of the process is the limit of the lengths of the curves at each step,

$$\lim_{n \to \infty} 3 \left(\frac{4}{3}\right)^n = \infty ,$$

since we have a geometric sequence with common ratio $r = \frac{4}{3} > 1$. ■

3. What is the area of the shape at step n of the process? *[2]*

SOLUTION. Using the analysis in the solution to question **1** above, at step $n-1$ of the process we have $3 \cdot 4^{n-1}$ line segments, and at step n of the process each line segment has length $\frac{1}{3^n}$, so each little triangle added at step n has area $\left(\frac{1}{3^n}\right)^2 = \frac{1}{3^{2n}} = \frac{1}{9^n}$ of the area of the original triangle, *i.e.* $\frac{1}{9^n} \cdot \frac{\sqrt{3}}{4}$. Hence the area *added* to the shape at step $n \geq 1$ is:

$$\begin{aligned}
&(\#\text{ triangles added at step } n) \times (\text{area of each triangle}) \\
&= (\#\text{ line segments at step } n-1) \times (\text{length of side})^2 \cdot (\text{area of original triangle}) \\
&= 3 \cdot 4^{n-1} \times \left(\left(\frac{1}{3}\right)^n\right)^2 \cdot \frac{\sqrt{3}}{4} = 3 \cdot \frac{4^n}{4} \times \frac{1}{9^4} \cdot \frac{\sqrt{3}}{4} = \left(\frac{4}{9}\right)^n \cdot \frac{3\sqrt{3}}{16} = \left(\frac{4}{9}\right)^{n-1} \cdot \frac{\sqrt{3}}{12}
\end{aligned}$$

It follows that the total area of the shape at step n is

$$\begin{aligned}
&\frac{\sqrt{3}}{4} + \left(\frac{4}{9}\right)^0 \cdot \frac{\sqrt{3}}{12} + \left(\frac{4}{9}\right)^1 \cdot \frac{\sqrt{3}}{12} + \cdots + \left(\frac{4}{9}\right)^{n-1} \cdot \frac{\sqrt{3}}{12} \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \left(1 + \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \cdots + \left(\frac{4}{9}\right)^{n-1}\right) \\
&= \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \frac{1 - \left(\frac{4}{9}\right)^n}{1 - \frac{4}{9}} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} \cdot \frac{9}{5} \cdot \left[1 - \left(\frac{4}{9}\right)^n\right] \\
&= \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} \cdot \left[1 - \left(\frac{4}{9}\right)^n\right] ,
\end{aligned}$$

with a bit of help from the formula for the sum of a finite geometric series. ■

4. Find the area of the shape after all the infinitely many steps of the process. *[3]*

SOLUTION. The area of the shape after all the infinitely many steps of the process is the limit of the areas of the shapes at each stage of the process:

$$\lim_{n \to \infty} \left(\frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} \cdot \left[1 - \left(\frac{4}{9}\right)^n\right]\right) = \frac{\sqrt{3}}{4} + \frac{3\sqrt{3}}{20} \cdot [1 - 0] = \frac{5\sqrt{3}}{20} + \frac{3\sqrt{3}}{20} = \frac{8\sqrt{3}}{20} = \frac{2\sqrt{3}}{5}$$

This uses the fact that $\lim_{n \to \infty} \left(\frac{4}{9}\right)^n = 0$, since $\left(\frac{4}{9}\right)^n$ is a geometric sequence with common ratio $r = \frac{4}{9} < 1$. ■