Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Winter 2012

Quiz Solutions

Quiz #1. ~~Monday, 16~~ Thursday, 19 January, 2012. *[10 minutes]*

1. Suppose $X \subset \mathbb{R}$ has $\sup(X) = \text{lub}(X) = a$. Show that if $Y = \{ -x \mid x \in X \}$, then $\inf(Y) = \text{glb}(Y) = -a$. *[5]*

Solution. First, we check that $-a$ is indeed a lower bound for Y:

If $y \in Y$, then, by the definition of Y, $y = -x$ for some $x \in X$. Since a is an upper bound for X, $a \geq x$, so $-a \leq -x = y$. Since this argument works for every $y \in Y$, it follows that $-a$ is a lower bound for Y.

Second, we check that $-a$ is also the greatest lower bound for Y:

Suppose e is any lower bound for Y, *i.e.* $e \leq y$ for all $y \in Y$. Since $Y = \{ -x \mid x \in X \}$, it follows that $e \leq -x$ for all $x \in X$, from which it follows that $-e \geq x$ for all $x \in X$. This means that $-e$ is an upper bound for X, so $-e \geq a$, because a is the least upper bound of X. But then $e \leq -a$. Thus $-a$ is the greatest lower bound of Y.

That's all, folks! ■

Quiz #2. Monday, 23 January, 2012. *[10 minutes]*

1. Suppose you are given that $\lim_{n\to\infty} \frac{1}{n} = 0$. Use this fact, plus some algebra and the limit laws for sequences, to compute $\lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n+2}$.

Solution. Here goes. Algebra first,

$$\lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n+2} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^2+2n+2} \cdot \frac{1/n^2}{1/n^2} = \lim_{n\to\infty} \frac{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{1}{n^2}}{\frac{n^2}{n^2}+\frac{2n}{n^2}+\frac{2}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1+\frac{2}{n}+\frac{2}{n^2}} = \lim_{n\to\infty} \frac{1+2\cdot\frac{1}{n}+\left(\frac{1}{n}\right)^2}{1+2\cdot\frac{1}{n}+2\cdot\left(\frac{1}{n}\right)^2}$$

... and then the limit laws:

$$= \frac{\lim_{n\to\infty}\left[1+2\cdot\frac{1}{n}+\left(\frac{1}{n}\right)^2\right]}{\lim_{n\to\infty}\left[1+2\cdot\frac{1}{n}+2\cdot\left(\frac{1}{n}\right)^2\right]} \quad \text{[Quotient Law]}$$

$$= \frac{\left[\lim_{n\to\infty} 1\right]+\left[\lim_{n\to\infty} 2\cdot\frac{1}{n}\right]+\left[\lim_{n\to\infty}\left(\frac{1}{n}\right)^2\right]}{\left[\lim_{n\to\infty} 1\right]+\left[\lim_{n\to\infty} 2\cdot\frac{1}{n}\right]+\left[\lim_{n\to\infty} 2\cdot\left(\frac{1}{n}\right)^2\right]} \quad \text{[Sum Law]}$$

$$= \frac{\left[\lim_{n\to\infty} 1\right]+2\left[\lim_{n\to\infty}\frac{1}{n}\right]+\left[\lim_{n\to\infty}\frac{1}{n}\right]^2}{\left[\lim_{n\to\infty} 1\right]+2\left[\lim_{n\to\infty}\frac{1}{n}\right]+2\left[\lim_{n\to\infty}\frac{1}{n}\right]^2} \quad \text{[Constant \& Product Laws]}$$

$$= \frac{1+2\cdot 0+0^2}{1+2\cdot 0+2\cdot 0^2} = \frac{1}{1} = 1 \quad \text{[Given fact]} \quad ■$$

Quiz #3. Monday, 30 January, 2012. *[10 minutes]*

1. If n is a positive integer, then the *square-free part of* n is $\upsilon(n) = \dfrac{n}{m^2}$, where m is the largest positive integer whose square divides n. Let $a_n = \dfrac{1}{\upsilon(n)}$ for $n \geq 1$. Find two subsequences of a_n which converge to different limits. *[5]*

Hint: The first thirty elements of the sequence are:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$\upsilon(n)$	1	2	3	1	5	6	7	2	1	10	11	3	13	14	15
a_n	1	$\frac{1}{2}$	$\frac{1}{3}$	1	$\frac{1}{5}$	$\frac{1}{6}$	$\frac{1}{7}$	$\frac{1}{2}$	1	$\frac{1}{10}$	$\frac{1}{11}$	$\frac{1}{3}$	$\frac{1}{13}$	$\frac{1}{14}$	$\frac{1}{15}$

n	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
$\upsilon(n)$	1	17	2	19	5	21	22	23	6	1	26	3	28	29	30
a_n	1	$\frac{1}{17}$	$\frac{1}{2}$	$\frac{1}{19}$	$\frac{1}{5}$	$\frac{1}{21}$	$\frac{1}{22}$	$\frac{1}{23}$	$\frac{1}{6}$	1	$\frac{1}{26}$	$\frac{1}{3}$	$\frac{1}{7}$	$\frac{1}{29}$	$\frac{1}{30}$

Solution. Note that since $1 \leq \upsilon(n) \leq n$ for each $n \geq 1$, we must always have $0 < a_n \leq 1$, so yhe sequence is bounded, and hence must have a convergent subsequence.

Here is a recipe that will find infinitely many subsequences of a_n, each with a different limit. For each integer $t > 0$ such that the largest m for which m^2 divides t is 1, *i.e.* for which $\upsilon(t) = t$, consider the subsequence a_{n_k} (for $k \geq 1$) of a_n given by $a_{n_k} = a_{tk^2} = \dfrac{1}{\upsilon\left(tk^2\right)}$. Since $\upsilon\left(tk^2\right) = t$, $a_{n_k} = \dfrac{1}{t}$ for each $k \geq 1$, so $\lim_{k\to\infty} a_{n_k} = \lim_{k\to\infty} \dfrac{1}{t} = \dfrac{1}{t}$.

For example, $t = 1$ gives the subsequence $a_1 = 1$, $a_4 = 1$, $a_9 = 1$, $a_{16} = 1$, ..., while $t = 2$ gives the subsequence $a_2 = \frac{1}{2}$, $a_8 = \frac{1}{2}$, $a_{18} = \frac{1}{2}$, $a_{32} = \frac{1}{2}$, ...

There are also lots of ways of picking subsequences that have a limit of 0. For example, if p_n is the nth prime number, then $a_{p_n} = \dfrac{1}{p_n}$. This gives the subsequence $a_2 = \frac{1}{2}$, $a_3 = \frac{1}{3}$, $a_5 = \frac{1}{5}$, $a_7 = \frac{1}{7}$, $a_{11} = \frac{1}{11}$, $a_{13} = \frac{1}{13}$, $a_{17} = \frac{1}{17}$, ... ■

Quiz #4. Monday, 6 February, 2012. *[10 minutes]*

1. Determine whether $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2 - n + 1}$ converges or not. *[5]*

Solution. We will use the Limit Comparison Test to show that the given series diverges because the Harmonic Series does so:

$$\lim_{n\to\infty} \frac{\frac{n}{n^2-n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n^2-n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^2}{n^2-n+1} = \lim_{n\to\infty} \frac{n^2}{n^2-n+1} \cdot \frac{1/n^2}{1/n^2}$$

$$= \lim_{n\to\infty} \frac{\frac{n^2}{n^2}}{\frac{n^2}{n^2} - \frac{n}{n^2} + \frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{n} + \frac{1}{n^2}} = \frac{1}{1-0+0} = 1$$

because $\dfrac{1}{n} \to 0$ and $\dfrac{1}{n^2} \to 0$ as $n \to \infty$. Since $0 < 1 < \infty$ and $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, it follows by the Limit Comparison Test that $\displaystyle\sum_{n=0}^{\infty} \frac{n}{n^2-n+1}$ does not converge either. ■

Quiz #5. Monday, 13 February, 2012. *[10 minutes]*

1. Determine whether $\displaystyle\sum_{n=0}^{\infty} \frac{n^2+n}{3^n}$ converges or not. *[5]*

SOLUTION. We will us the Ratio Test to show that the given series converges. Note that all terms of this series are ≥ 0.

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)^2+(n+1)}{3^{n+1}}}{\frac{n^2+n}{3^n}} \right| = \lim_{n\to\infty} \frac{(n+1)^2+(n+1)}{n^2+n} \cdot \frac{3^n}{3^{n+1}} = \lim_{n\to\infty} \frac{(n+1)\left((n+1)+1\right)}{n(n+1)} \cdot \frac{1}{3}$$
$$= \frac{1}{3} \lim_{n\to\infty} \frac{n+2}{n} = \frac{1}{3} \lim_{n\to\infty} \left(\frac{n}{n} + \frac{2}{n}\right) = \frac{1}{3} \lim_{n\to\infty} \left(1 + \frac{2}{n}\right)$$
$$= \frac{1}{3}(1+0) = \frac{1}{3} < 1$$

Since the limit gives a value less than 1, it follows by the Ratio Test that the given series converges. ■

Quiz #6. Monday, 27 February, 2012. *[10 minutes]*

1. Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} x^n . \quad [5]$$

SOLUTION. We will use the Ratio Test to find the radius of convergence.

$$\lim_{n\to\infty} \left| \frac{\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)\left(\frac{1}{2}-(n+1)-1\right)}{(n+1)!} x^{n+1}}{\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-n+1\right)}{n!} x^n} \right|$$
$$= \lim_{n\to\infty} \left| \frac{\frac{1}{2}-n}{n+1} x \right| = \lim_{n\to\infty} \frac{n-\frac{1}{2}}{n+1} |x| = 1 \cdot |x| = |x|$$

It follows by the Ratio Test that the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$, *i.e.* its radius of convergence is $R = 1$.

To find the interval of convergence we need to determine whether the series converges or not at the endpoints, $x = \pm 1$. We will apply Gauss' Test – check out the handout about it! Observe that the absolute value of the ratio of coefficients in the computation for the Ratio Test above came down to $\frac{n-\frac{1}{2}}{n+1}$. The highest power of n appearing is 1 and is the same in both the numerator and denominator, where it occurs with coefficient 1 in both, so we need not work further to make them so. Looking at the coefficients of the next lowest power of n, *i.e.* the constant terms, we observe that $-\frac{1}{2} < 0 = 1 - 1$. It follows by part 5 of Gauss' Test that the series is absolutely convergent when $x = \pm R = \pm 1$, *i.e.* it converges at both endpoints. The interval of convergence of the given series is therefore $[-1, 1]$. ■

Quiz #7. Monday, 5 March, 2012. *[10 minutes]*

1. Verify that the sequence of functions $f_n(x) = e^{-nx}$ converges uniformly to the function $f(x) = 0$ on the interval $[1, 2]$. *[5]*

SOLUTION. We need to check that for any $\varepsilon > 0$, there is an N such that if $n \geq N$, then $|f_n(x) - f(x)| = |e^{-nx} - 0| < \varepsilon$ for all $x \in [1, 2]$.

Suppose an $\varepsilon > 0$ is given. Then, recalling that $e^{-t} > 0$ for all t,

$$\left|e^{-nx} - 0\right| < \varepsilon \quad \iff \quad e^{-nx} < \varepsilon \,.$$

Since e^{-t} is a decreasing function, $e^{-n} = e^{-n1} \geq e^{-nx}$ for all $x \in [1, 2]$. Thus it will suffice to ensure that $e^{-n} < \varepsilon$. Note that

$$e^{-n} < \varepsilon \quad \iff \quad -n < \ln(\varepsilon) \quad \iff \quad n > -\ln(\varepsilon) \quad \iff \quad n > \ln(1/\varepsilon)\ .$$

Let $N > \ln(1/\varepsilon)$. Then if $n \geq N > \ln(1/\varepsilon)$, we have $|e^{-nx} - 0| = e^{-nx} \leq e^{-n} < \varepsilon$ for all $x \in [1, 2]$.

Thus $f_n(x) = e^{-nx}$ converges uniformly to $f(x) = 0$ on the interval $[1, 2]$. ∎

Quiz #8. Monday, 12 March, 2012. *[15 minutes]*

1. Give an example of a sequence of continuous functions $f_n(x)$ defined on a closed interval $[a, b]$ such that $c_n = \int_a^b f_n(x)\,dx$ converges to some real number c, but $f_n(x)$ does not converge uniformly on $[a, b]$. *[5]*

SOLUTION. We'll go whole hog and find such a sequence such that $f_n(x)$ doesn't even converge pointwise for various $x \in [a, b]$.

Let $[a, b] = [-\pi, \pi]$ and $f_n(x) = sin(nx)$ for $n \geq 0$. Then each f_n is certainly defined and continuous on $[a, b]$.

When $n \geq 1$, the substitution $u = nx$, so $du = n\,dx$, and thus $dx = \frac{1}{n}\,du$ and $\begin{smallmatrix} x & -\pi & \pi \\ u & -n\pi & n\pi \end{smallmatrix}$, gives:

$$\begin{aligned} \int_{-\pi}^{\pi} \sin(nx)\,dx &= \int_{-n\pi}^{n\pi} \sin(u) \cdot \frac{1}{n}\,du = \frac{1}{n}\cos(u)\Big|_{-n\pi}^{n\pi} \\ &= \frac{1}{n}\cos(n\pi) - \frac{1}{n}\cos(-n\pi) = \frac{1}{n}\cos(n\pi) - \frac{1}{n}\cos(n\pi) = 0 \end{aligned}$$

[Recall that $\cos(-x) = \cos(x)$ for all x.] The special case $n = 0$ is left to the reader ... Thus it is certainly the case that $c_n = \int_{-\pi}^{\pi} \sin(nx)\,dx = 0$ converges. (To? :-)

However, $f_n(x) = \sin(nx)$ does not even converge pointwise for some $x \in [-\pi, \pi]$. For example, consider $x = \pi/2$. If n is even, then $\sin(n\pi/2) = 0$, but if n is odd, then $\sin(n\pi/2) = \pm 1$, depending upon whether $n = 1 \pmod 4$ or $n = 3 \pmod 4$. Since there are infinitely many even and infinitely many odd numbers of each type, it follows that $\lim_{n\to\infty} \sin(n\pi/2)$ doesn't exist. Given that $f_n(x)$ doesn't even converge pointwise for some x in the interval, it cannot converge uniformly. ∎

Quiz #9. Monday, 19 March, 2012. *[10 minutes]*

1. Find the radius of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$. *[2]*

2. Assuming that $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ when the series converges, find a power series equal to $\cos(x)$. *[3]*

SOLUTION TO 1. As usual, we use the Ratio Test to find the radius of convergence:

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(2(n+1)+1)!} x^{2(n+1)+1}}{\frac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(2n+3)!} x^{2n+3}}{\frac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(-1)^{n+1} x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n x^{2n+1}} \right| = \lim_{n\to\infty} \left| \frac{-1}{(2n+2)(2n+3)} x^2 \right| \\
&= \left| x^2 \right| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} = \left| x^2 \right| \cdot 0 = 0 \,,
\end{aligned}$$

since $(2n+2)(2n+3) \to \infty$ as $n \to \infty$. It follows by the Ratio Test that the series converges no matter what the value of x is, *i.e.* the radius of convergence is $R = \infty$. □

SOLUTION TO 2. Within the radius of convergence of a power series we can differentiate or integrate it term-by-term and get the power series that adds up to the derivative or integral, respectively, of the function that the original power series added up to. In this case, since $\cos(x) = \frac{d}{dx} \sin(x)$, we get:

$$\begin{aligned}
\cos(x) = \frac{d}{dx} \sin(x) &= \frac{d}{dx} \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \right) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}
\end{aligned}$$

Note that this series must – and actually does! – have the same radius of convergence as the original power series for $\sin(x)$, *i.e.* $R = \infty$. ■