Mathematics 3790H – Analysis I: Introduction to analysis
Trent University, Fall 2010

Solutions to Assignment #2

Cesàro Salad?

Ernesto Cesàro (1859–1906) was an Italian mathematician who worked in the field of differential geometry. Along the way he came up with some interesting ideas about the convergence of sequences and series.

1. A sequence $\{t_n\}$ is said to be *Cesàro-summable* if $\displaystyle\lim_{n\to\infty} \frac{t_1 + t_2 + t_3 + \cdots + t_n}{n}$ exists. Show that any convergent sequence is Cesàro-summable. *[4]*

Solution. Suppose $\displaystyle\lim_{n\to\infty} t_n = \tau$. We will show that $\displaystyle\lim_{n\to\infty} \frac{t_1 + t_2 + t_3 + \cdots + t_n}{n} = \tau$ as well.

Suppose $\varepsilon > 0$ is given. Since $\displaystyle\lim_{n\to\infty} t_n = \tau$, there is an N such that for all $n \geq N$, $|t_n - \tau| < \frac{\varepsilon}{3}$. Choose $M > N$ such that both $\left|\frac{t_1 + t_2 + t_3 + \cdots + t_{N-1}}{M}\right| < \frac{\varepsilon}{3}$ and $\left|\frac{(N-1)\,\tau}{M}\right| < \frac{\varepsilon}{3}$. Now suppose $m \geq M > N$. Then

$$\begin{aligned}
&\left|\frac{t_1 + t_2 + t_3 + \cdots + t_m}{m} - \tau\right| \\
&= \left|\frac{t_1 + t_2 + t_3 + \cdots + t_{N-1}}{m} + \frac{t_N + t_{N+1} + \cdots + t_m}{m} - \tau\right| \\
&\leq \left|\frac{t_1 + t_2 + t_3 + \cdots + t_{N-1}}{m}\right| + \left|\frac{t_N + t_{N+1} + \cdots + t_m}{m} - \tau\right| \\
&< \frac{\varepsilon}{3} + \left|\frac{t_N + t_{N+1} + \cdots + t_m - m\tau}{m}\right| \\
&= \frac{\varepsilon}{3} + \left|\frac{(t_N - \tau) + (t_{N+1} - \tau) + \cdots + (t_m - \tau) + (N-1)\,\tau}{m}\right| \\
&\leq \frac{\varepsilon}{3} + \frac{|t_N - \tau| + |t_{N+1} - \tau| + \cdots + |t_m - \tau|}{m} + \left|\frac{(N-1)\,\tau}{m}\right| \\
&< \frac{\varepsilon}{3} + \frac{(m - N + 1)}{m} \cdot \frac{\varepsilon}{3} + \left|\frac{(N-1)\,\tau}{M}\right| \\
&< \frac{\varepsilon}{3} + \frac{m}{m} \cdot \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.
\end{aligned}$$

Thus, by the $\varepsilon - N$ definition of limits, $\displaystyle\lim_{n\to\infty} \frac{t_1 + t_2 + t_3 + \cdots + t_n}{n} = \tau$, as desired. ■

Bonus: Is a Cesàro-summable sequence necessarily convergent? Prove it is or give a counterexample. *[1]*

SOLUTION. A Cesàro-summable sequence is not necessarily convergent. For a cheap counterexample, consider the sequence $t_n = (-1)^{n+1}$, *i.e.* $1, -1, 1, -1, \ldots$ This sequence does not converge, since the terms alternate between -1 and 1, but $\lim_{n\to\infty} \frac{1-1+1-\cdots+(-1)^{n+1}}{n} = 0$ because $0 \le \frac{1-1+1-\cdots+(-1)^{n+1}}{n} \le \frac{1}{n}$. ■

For questions **2** and **3**, you may assume that the following result is true:

Stolz–Cesàro Theorem. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $\{b_n\}$ is increasing, $b_n > 0$ for all n, $\lim_{n\to\infty} b_n = \infty$, and $\lim_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n}$ exists or is equal to $\pm\infty$. Then $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{a_{n+1} - a_n}{b_{n+1} - b_n}$.

This theorem is in some measure a generalization both of the notion of Cesàro summation (see **2** below) and of l'Hôpital's Rule.

2. Let $p \in \mathbb{R}$, $p \neq -1$. Using the Stolz–Cesàro Theorem, compute the limit

$$\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}}. \quad [3]$$

SOLUTION. First, suppose $p > -1$, so $p + 1 > 0$ and $\lim_{n\to\infty} n^{p+1} = \infty$. We will apply the Stolz–Cesàro Theorem with $a_n = 1^p + 2^p + \ldots + n^p$ and $b_n = n^{p+1}$:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} &= \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} \\
&= \lim_{n\to\infty} \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}} \cdot \frac{1/(n+1)^p}{1/(n+1)^p} \\
&= \lim_{n\to\infty} \frac{1}{(n+1) - n\left(\frac{n}{n+1}\right)^p} \\
&= \lim_{n\to\infty} \frac{1}{(n+1) - n\left(1 + \frac{-1}{n+1}\right)^p}
\end{aligned}$$

Here we use the Generalized Binomial Theorem to expand the remaining power:

$$\left(1 - \frac{1}{n+1}\right)^p = \sum_{k=0}^{\infty} \binom{p}{k} \frac{(-1)^k}{(n+1)^k} = 1 - \frac{p}{1!} \cdot \frac{1}{n+1} + \frac{p(p-1)}{2!} \cdot \frac{1}{(n+1)^2} - \cdots$$

Thus

$$
\begin{aligned}
\lim_{n\to\infty} \frac{1^p + 2^p + \cdots + n^p}{n^{p+1}} &= \lim_{n\to\infty} \frac{1}{(n+1) - n\left(1 + \frac{-1}{n+1}\right)^p} \\
&= \lim_{n\to\infty} \frac{1}{(n+1) - n\left(1 - \frac{p}{1!}\cdot\frac{1}{n+1} + \frac{p(p-1)}{2!}\cdot\frac{1}{(n+1)^2} - \cdots\right)} \\
&= \lim_{n\to\infty} \frac{1}{n+1-n+p\cdot\frac{n}{n+1} - \frac{p(p-1)}{2}\cdot\frac{n}{(n+1)^2} + \cdots} \\
&= \frac{1}{\lim_{n\to\infty}\left(1 + p\cdot\frac{n}{n+1} - \frac{p(p-1)}{2}\cdot\frac{n}{(n+1)^2} + \cdots\right)} \\
&= \frac{1}{1 + p\cdot 1 - \frac{p(p-1)}{2}\cdot 0 + \cdots} \\
&= \frac{1}{1+p-0+0-\cdots} = \frac{1}{p+1},
\end{aligned}
$$

so we have computed the desired limit, at least when $p > -1$. Whew! (We swept a little something under the rug along the way. What was it?)

Second, suppose $p < -1$, so $p + 1 < 0$. In this case, $\lim_{n\to\infty} n^{p+1} = 0 \neq \infty$ (Why?), so we can't apply the Stolz–Cesàro Theorem as we did in the previous case. Moreover, $\frac{1}{p+1}$ can't be the limit in this case because it is negative when $p < -1$, and $\frac{1^p+2^p+\ldots+n^p}{n^{p+1}} > 0$ for all $n \geq 1$. In fact, when $p < -1$, $\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} = \infty$. To see this, observe that

$$\lim_{n\to\infty} \left(1^p + 2^p + \ldots + n^p\right) = \lim_{n\to\infty} \left(\frac{1}{1^{|p|}} + \frac{1}{2^{|p|}} + \cdots + \frac{1}{n^{|p|}}\right)$$

exists when $p < -1$, because the series $\sum_{k=1}^{\infty} \frac{1}{k^{|p|}}$ converges (to some positive number) by the p-Test when $|p| > 1$. On the other hand, since $\lim_{n\to\infty} n^{p+1} = 0$, we must have $\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}} = \infty$. ∎

Bonus: Use **2** to compute $\lim_{n\to\infty} \sqrt[n]{n!}$. *[1]*

Solution. I'm too lazy to write it out, but what you really want is to use the method used to do **3** below. ∎

3. Let $\{c_n\}$ be a sequence of positive real numbers. Use the Stolz–Cesàro Theorem to show that if $\lim_{n\to\infty} \frac{c_{n+1}}{c_n}$ exists or is $\pm\infty$, then $\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$. *[4]*

SOLUTION. First, a small sanity check: if $\{c_n\}$ is a sequence of positive real numbers, $\lim_{n\to\infty} \frac{c_{n+1}}{c_n}$ can't be $-\infty$, so we only need to consider the possibilities that the limit exists (and is ≥ 0 to boot) or is $+\infty$.

Suppose, then, that $\lim_{n\to\infty} \frac{c_{n+1}}{c_n}$ exists or is $+\infty$. Then

$$\ln\left(\lim_{n\to\infty} \frac{c_{n+1}}{c_n}\right) = \lim_{n\to\infty} \ln\left(\frac{c_{n+1}}{c_n}\right)$$

since $\ln(x)$ is an increasing and continuous function for $x > 0$. We will show that

$$\lim_{n\to\infty} \ln\left(\frac{c_{n+1}}{c_n}\right) = \lim_{n\to\infty} \ln\left(\sqrt[n]{c_n}\right)$$

with the help of the Stolz–Cesàro Theorem.

Let $a_n = \ln(c_n)$ and $b_n = n$. Then b_n clearly satisfies the required hypotheses in the Stolz–Cesàro Theorem, so

$$\begin{aligned}
\lim_{n\to\infty} \frac{\ln(c_n)}{n} &= \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} \\
&= \lim_{n\to\infty} \frac{\ln(c_{n+1}) - \ln(c_n)}{(n+1) - n} \\
&= \lim_{n\to\infty} \frac{\ln(c_{n+1}/c_n)}{1} \\
&= \lim_{n\to\infty} \frac{c_{n+1}}{c_n}\,.
\end{aligned}$$

It follows that

$$\ln\left(\lim_{n\to\infty} \frac{c_{n+1}}{c_n}\right) = \lim_{n\to\infty} \ln\left(\frac{c_{n+1}}{c_n}\right) = \lim_{n\to\infty} \frac{\ln(c_n)}{n} = \lim_{n\to\infty} \ln\left(\sqrt[n]{c_n}\right)\,,$$

so we must have $\lim_{n\to\infty} \frac{c_{n+1}}{c_n} = \lim_{n\to\infty} \sqrt[n]{c_n}$, as desired. ■