Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2010

Take-home Final Exam
Due: Wednesday, 22 December, 2010

Instructions: Do all three of parts **I** – **III**, and, if you wish, part ↻ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem, and use calculators or computer software to do numerical computations and to check your algebra. However, you may not consult any other sources, nor consult or work with any other person on this exam.

Part I. Do all *three* of problems **1** – **3**. *[30 = 3 × 10 each]*

1. Suppose that $\sum_{n=0}^{\infty} a_n$ converges absolutely. Show that $\sum_{n=0}^{\infty} a_n^3$ converges absolutely too.

2. Use the $\varepsilon - \delta$ definition of limits to show that $\lim_{t \to 1} \left(t^2 - 2t + 1\right) = 0$.

3. Determine the radius and interval of convergence of the power series

$$1 + 2x + 6x^2 + 20x^3 + \cdots = \sum_{k=0}^{\infty} \frac{(2k)!}{k! \cdot k!} x^k .$$

Part II. Do any *two* of problems **4** – **7**. *[20 = 2 × 10 each]*

4. Show that $f(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is continuous at $x = 0$, but not at any other point.

5. How many terms of the Taylor series for $\ln(x+1)$ about 0, evaluated at $x = 1$, are needed to guarantee that the partial sum is within 0.01 of $\ln(2) = \ln(1+1)$?

6. Suppose $\sum_{i=0}^{\infty} a_i x^i$ is a power series with interval of convergence $[-1, 1]$, such that $a_i \geq 0$ for all $i \geq 0$. Show that $\sum_{i=0}^{\infty} a_i x^i$ converges uniformly on $(-1, 1)$.

7. Give an example of two power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ which both converge for all x and such that $\left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{n=0}^{\infty} b_n x^n\right) = 1$ for all x, or show that such a pair of series cannot exist.

[Parts **III** – ↑↓↑↓ on page 2.]

Part III. Do any *two* of problems **8** – **11**. *[20 = 2 × 10 each]*

8. Suppose a_k is a sequence of positive real numbers such that $a_{k+1} \leq a_k$ for all k and $\sum_{n=0}^{\infty} 2^n a_{2^n}$ converges. Show that $\sum_{k=1}^{\infty} a_k$ converges.

9. Suppose that (a, b) is an open interval and that for each $k \geq 0$, $f_k(x)$ is a function on (a, b) and M_k is a real number such that $|f_k(x)| \leq M_k$ for all x in (a, b). Show that if $\sum_{k=0}^{\infty} M_k$ converges, then $F_n(x) = \sum_{k=0}^{n} f_k(x)$ converges uniformly on (a, b).

10. Find a sequence of positive real numbers a_k for $k \geq 0$ such that

$$\lim_{m \to \infty} \left(a_0^m + a_1^m + a_2^m + \cdots + a_m^m\right) = 1\,.$$

11. $f(x)$ is *uniformly continuous* on an interval I if for any $\varepsilon > 0$ there is a $\delta > 0$ such that for any points x and z in I, if $|x - z| < \delta$, then $|f(x) - f(z)| < \varepsilon$. Show that if $f(x)$ is differentiable on a closed and bounded interval $[a, b]$ and $f'(x) \leq C$ for some constant C on $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$.

Part ↑↓↑↓.

⇜⇝. Write a poem about real analysis or mathematics in general. *[2]*

[Total = 70]

HAVE A GOOD BREAK!!