Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2010

Assignment #4
Due on Thursday, 11 November, 2010.

The integral form of the remainder of a Taylor series*

In what follows, let us suppose that c is a real number and $f(x)$ is a function such that $f^{(n)}(x)$ is defined and continuous for all $n \geq 0$ and for all x in some open interval I containing c. Recall that for $n \geq 0$, the Taylor polynomial of degree n of $f(x)$ at c is

$$\begin{aligned} T_n(x) &= f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\ &= \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k \,, \end{aligned}$$

and that the corresponding remainder term is $R_n(x) = f(x) - T_n(x)$. In what follows, we will assume that every x we encounter is in the interval I.

1. Use the Fundamental Theorem of Calculus to show that

$$R_0(x) = \int_c^x f'(t)\,dt \,. \quad [1]$$

2. Use the formula in **1** and integration by parts to show that

$$R_1(x) = \int_c^x f''(t)(x-t)\,dt \,. \quad [2]$$

Hint: Use the parts $u = f'(t)$ and $v = t - x$ ...

3. Use the formula in **2** and integration by parts to show that

$$R_2(x) = \frac{1}{2}\int_c^x f^{(3)}(t)(x-t)^2\,dt \,. \quad [2]$$

4. Use induction to show that

$$R_n(x) = \frac{1}{n!}\int_c^x f^{(n+1)}(t)(x-t)^n\,dt \,. \quad [5]$$

* Theorem 7.45 in the text.