Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2010

Assignment #2
Due on Thursday, 7 October, 2010.

Cesàro Salad?

Ernesto Cesàro (1859–1906) was an Italian mathematician who worked in the field of differential geometry. Along the way he came up with some interesting ideas about the convergence of sequences and series.

1. A sequence $\{t_n\}$ is said to be *Cesàro-summable* if $\displaystyle\lim_{n\to\infty} \frac{t_1 + t_2 + t_3 + \cdots + t_n}{n}$ exists. Show that any convergent sequence is Cesàro-summable. *[4]*

Bonus: Is a Cesàro-summable sequence necessarily convergent? Prove it is or give a counterexample. *[1]*

For questions **2** and **3**, you may assume that the following result is true:

Stolz–Cesàro Theorem. Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers such that $\{b_n\}$ is increasing, $b_n > 0$ for all n, $\displaystyle\lim_{n\to\infty} b_n = \infty$, and $\displaystyle\lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$ exists or is equal to $\pm\infty$. Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n}$.

This theorem is in some measure a generalization both of the notion of Cesàro summation (see **2** below) and of l'Hôpital's Rule.

2. Let $p \in \mathbb{R}$, $p \neq -1$. Using the Stolz–Cesàro Theorem, compute the limit

$$\lim_{n\to\infty} \frac{1^p + 2^p + \ldots + n^p}{n^{p+1}}. \quad [3]$$

Bonus: Use **2** to compute $\displaystyle\lim_{n\to\infty} \sqrt[n]{n!}$. *[1]*

3. Let $\{c_n\}$ be a sequence of positive real numbers. Use the Stolz–Cesàro Theorem to show that if $\displaystyle\lim_{n\to\infty} \frac{c_{n+1}}{c_n}$ exists or is $\pm\infty$, then $\displaystyle\lim_{n\to\infty} \sqrt[n]{c_n} = \lim_{n\to\infty} \frac{c_{n+1}}{c_n}$. *[4]*

NOTE: Compare **3** to the hypotheses of the Ratio and Root Tests for convergence of series.