Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2010

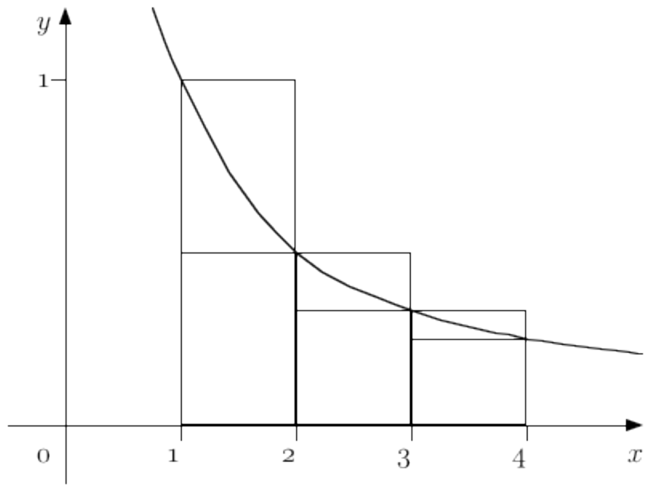
Assignment #1
Due on Thursday, 23 September, 2010.

Euler's Constant

Euler's constant* is the real number γ defined by:

$$\gamma = \lim_{n\to\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln(n) \right] = \lim_{n\to\infty} \left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} - \ln(n) \right]$$

Since $\ln(n) = \int_1^n \frac{1}{x}\, dx$, we can think of γ as a sum of areas: for each $k \geq 1$, consider the area of the rectangle of height $\frac{1}{k}$ with base the interval $[k, k+1]$ with the part below the curve $y = \frac{1}{x}$ taken away.



Your task is to show that the definition of Euler's constant makes sense.

1. Show that $\lim_{n\to\infty} \left[\left(\sum_{k=1}^{n} \frac{1}{k} \right) - \ln(n) \right]$ exists. *[7]*

Hint: For each rectangle of height $\frac{1}{k}$ with base the interval $[k, k+1]$ take away the part that lies below $y = \frac{1}{k+1}$.

2. Show that $\frac{1}{2} \leq \gamma \leq 1$. *[3]*

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* It is traditionally denoted by γ and sometimes called the Euler-Mascheroni constant. In case you're curious, $\gamma = 0.5772156649\ldots$ It is unknown whether γ is rational or not.