Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2009

Solutions to Assignment #6

Suppose α, β, and γ are any real numbers not in $\mathbb{Z}^{\leq 0} = \{0, -1, -2, \dots\}$, and consider the following power series:

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)} x^3 + \cdots$$
$$= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\dots(\alpha+n-1) \cdot \beta(\beta+1)\dots(\beta+n-1)}{n! \cdot \gamma(\gamma+1)\dots(\gamma+n-1)} x^n$$

This is what used to be called a hypergeometric series before the more general definition used in our textbook came along.

1. Why are the constants α, β, and γ not allowed to be 0 or any negative integer in the definition above? *[1]*

Solution. If γ were allowed to be 0 or a negative integer, then one would eventually be dividing by 0 in the expression $\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\cdot\beta(\beta+1)\dots(\beta+n-1)}{n!\cdot\gamma(\gamma+1)\dots(\gamma+n-1)}$ for the coefficient of x^n. On the other hand, if α or β were allowed to be 0 or a negative integer, then the coefficient of x^n would eventually always be 0, making the series a polynomial. This is, presumably, a little less interesting. ■

2. Show that Newton's binomial series is a series of this type. *[1]*

Solution. Newton's binomial series is

$$1 + ax + \frac{a(a-1)}{2!} x^2 + \frac{a(a-1)(a-2)}{3!} x^n + \cdots = 1 + \sum_{n=1}^{\infty} \frac{a(a-1)\cdots(a-n+1)}{n!} x^n ,$$

which converges absolutely to $(1+x)^a$ for $|x| < 1$ and diverges when $|x| > 1$. This is just a hypergeometric series with $\alpha = a$, and where $\beta = \gamma$ can be any number you like except a non-negative integer, so that $\frac{\beta(\beta+1)\dots(\beta+n-1)}{\gamma(\gamma+1)\dots(\gamma+n-1)} = 1$ for all $n \geq 1$. ■

3. Determine for which values of x this series respectively converges absolutely, converges conditionally, and diverges. *[9]*

Solution. We will use the Ratio Test to find the radius of convergence, R, of the series, and then, if $R < \infty$, examine the endpoints, $-R$ and R, of the interval of convergence to see what the series does there.

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{\frac{\alpha(\alpha+1)\dots(\alpha+n)\cdot\beta(\beta+1)\dots(\beta+n)}{(n+1)!\cdot\gamma(\gamma+1)\dots(\gamma+n)}x^{n+1}}{\frac{\alpha(\alpha+1)\dots(\alpha+n-1)\cdot\beta(\beta+1)\dots(\beta+n-1)}{n!\cdot\gamma(\gamma+1)\dots(\gamma+n-1)}x^{n}}\right| \\
&= \lim_{n\to\infty}\left|\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}x\right| = \lim_{n\to\infty}\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)}|x| \\
&\qquad\text{(Since } \tfrac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} > 0 \text{ once } n \text{ is large enough.)} \\
&= \lim_{n\to\infty}\frac{\alpha\beta+(\alpha+\beta)n+n^2}{\gamma+(\gamma+1)n+n^2}|x| = |x|\lim_{n\to\infty}\frac{\frac{\alpha\beta}{n^2}+\frac{\alpha+\beta}{n}+1}{\frac{\gamma}{n^2}+\frac{\gamma+1}{n}+1} \\
&= |x|\frac{0+0+1}{0+0+1} = |x|
\end{aligned}$$

It follows by the Ratio Test that the series converges absolutely when $|x| < 1$ and diverges when $|x| > 1$, so the radius of convergence of this series is $R = 1$.

It remains to determine whether, and how, the series converges at the endpoints of the interval of convergence, $x = -1$ and $x = 1$. Note that because $\frac{(\alpha+n)(\beta+n)}{(n+1)(\gamma+n)} > 0$ once n is large enough, the series eventually alternates when $x = -1$, but is eventually always positive when $x = 1$. We will use Gauss' Test to see what happens at these points, since we know from the work above that when $|x| = 1$,

$$\frac{a_{n+1}}{a_n} = \frac{\alpha\beta+(\alpha+\beta)n+n^2}{\gamma+(\gamma+1)n+n^2},$$

which is a ratio of monic polynomials of the same degree. The key is to compare the coefficients of the next-to-lowest powers of n in the numerator, $\alpha+\beta$, and the denominator, $\gamma+1$.

By parts 1 and 2 of Gauss' Test, the series diverges for $x = \pm 1$ when $\alpha+\beta \geq \gamma+1$; by parts 3 and 4, it converges (conditionally) at $x = -1$ but diverges at $x = 1$ when $\gamma = (\gamma+1)-1 \leq \alpha+\beta < \gamma+1$; and by part 5, it converges absolutely for $x = \pm 1$ when $\alpha+\beta < (\gamma+1)-1 = \gamma$. Whew! ■