Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2009

Solutions to Assignment #2

For questions **1** and **2**, assume that we know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

for all $x \in \mathbb{R}$.

1. Work out the power series for a^x, where a is a positive real number. *[3]*

SOLUTION. Note that if $a > 0$, then $a = e^{\ln(a)}$. It follows that

$$\begin{aligned}
a^x &= \left(e^{\ln(a)}\right)^x = e^{\ln(a)\cdot x} \\
&= 1 + \frac{\ln(a)\cdot x}{1!} + \frac{(\ln(a)\cdot x)^2}{2!} + \frac{(\ln(a)\cdot x)^3}{3!} + \cdots \\
&= 1 + \frac{\ln(a)}{1!}x + \frac{(\ln(a))^2}{2!}x^2 + \frac{(\ln(a))^3}{3!}x^3 + \cdots \\
&= \sum_{n=0}^{\infty} \frac{(\ln(a))^n}{n!} x^n . \qquad \blacksquare
\end{aligned}$$

2. Show that $e^{s+t} = e^s e^t$ by doing algebra with the appropriate power series. *[4]*

SOLUTION. One key to doing this one efficiently is to use the Binomial Theorem. Recall that if $q \geq 0$, then

$$(s+t)^q = s^q + qs^{q-1}t + \frac{q(q-1)}{2}s^{q-2}t^2 + \cdots + qst^{q-1} + t^q = \sum_{p=0}^{q} \frac{q!}{p!(q-p)!} s^{q-p}t^p .$$

It follows that

$$\frac{(s+t)^q}{q!} = \frac{1}{q!}\sum_{p=0}^{q} \frac{q!}{p!(q-p)!} s^{q-p}t^p = \sum_{p=0}^{q} \frac{1}{p!(q-p)!} s^{q-p}t^p .$$

Setting $n = q - p$ and $k = p$ in the last, allows us to rewrite this in the form we are going to need:

$$\frac{(s+t)^q}{q!} = \sum_{\substack{n,\, k \geq 0 \\ n+k=q}} \frac{s^n t^k}{n!k!} .$$

So equipped, off we go, using the distributive laws and the formula we derived above:

$$\begin{aligned}
e^s e^t &= \left(1 + \frac{s}{1!} + \frac{s^2}{2!} + \frac{s^3}{3!} + \frac{s^4}{4!} + \cdots\right)\left(1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right) \\
&= \left(\sum_{n=0}^{\infty} \frac{s^n}{n!}\right) \cdot \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} \frac{s^n}{n!} \left(\sum_{k=0}^{\infty} \frac{t^k}{k!}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{s^n}{n!} \cdot \frac{t^k}{k!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{s^n t^k}{n!k!}
\end{aligned}$$

Note that in this last sum, we are summing $\frac{s^n t^k}{n!k!}$ over all possible combinations of $n \geq 0$ and $k \geq 0$. We will list all these combinations a little differently by grouping them according to what the sum $n + k$ amounts to:

$$\begin{aligned}
&= \sum_{q=0}^{\infty} \left(\sum_{\substack{n,\, k \geq 0 \\ n+k=q}} \frac{s^n t^k}{n!k!} \right) \\
&= \sum_{q=0}^{\infty} \frac{(s+t)^q}{q!} \qquad \text{(Using the formula obtained previously.)} \\
&= 1 + \frac{s+t}{1!} + \frac{(s+t)^2}{2!} + \frac{(s+t)^3}{3!} + \frac{(s+t)^4}{4!} + \frac{(s+t)^5}{5!} \cdots \\
&= e^{s+t}
\end{aligned}$$

Whew! ■

3. The modern (and Archimedean!) meaning of "the series $\sum_{i=0}^{\infty} a_i$ converges to A" is usually captured by a definition like:

($*$) $\sum_{i=0}^{\infty} a_i$ converges to A if for every $\varepsilon > 0$ there is a K such that for all $k \geq K$ we have $\left| \left(\sum_{i=0}^{k} a_i \right) - A \right| < \varepsilon$.

Archimedes himself would probably have said something more along the following lines:

($\bullet$) $\sum_{i=0}^{\infty} a_i$ converges to A if both

(1) for every $L < A$ there is a K such that for all $k \geq K$ we have $L < \left(\sum_{i=0}^{k} a_i \right)$,

and

(2) for every $U > A$ there is a K' such that for all $k \geq K'$ we have $\left(\sum_{i=0}^{k} a_i \right) < U$.

Explain, in detail, why these two definitions are actually equivalent. *[3]*

Solution. We'll show that each statement implies the other separately. Suppose $\sum_{i=0}^{\infty} a_i$ is a series and A is a number.

($\Longrightarrow$) Assume $\sum_{i=0}^{\infty} a_i$ converges to A in the sense of $(*)$, and suppose $L < A$ and $U > A$ are given. Let $\varepsilon = \min(A - L, U - A)$.

By $(*)$, there is a K such that for all $k \geq K$ we have $\left|\left(\sum_{i=0}^{k} a_i\right) - A\right| < \varepsilon$. It follows that for all $k \geq K$ we have $A - \left(\sum_{i=0}^{k} a_i\right) < \varepsilon \leq A - L$, so $-\sum_{i=0}^{k} a_i < -L$, and hence $L < \sum_{i=0}^{k} a_i$.

Similarly, by $(*)$, there is a K such that for all $k \geq K$ we have $\left|\left(\sum_{i=0}^{k} a_i\right) - A\right| < \varepsilon$. It follows that for all $k \geq K' = K$ we have $\left(\sum_{i=0}^{k} a_i\right) - A < \varepsilon \leq U - A$, so $\sum_{i=0}^{k} a_i < U$.

Since both parts of $(\bullet)$ are satisfied, $\sum_{i=0}^{\infty} a_i$ converges to A in the sense of $(\bullet)$.

($\Longleftarrow$) Assume $\sum_{i=0}^{\infty} a_i$ converges to A in the sense of $(\bullet)$, and suppose $\varepsilon > 0$ is given. Let $L = A - \varepsilon$ and $U = A + \varepsilon$; note that $L < A < U$.

By part (1) of $(\bullet)$, there is a K such that for all $k \geq K$ we have $L < \sum_{i=0}^{k} a_i$, and, by part (2), there is a K' such that for all $k \geq K'$ we have $\sum_{i=0}^{k} a_i < U$. Let $N = \max(K, K`)$. Then, for all $k \geq N$, we have

$$A - \varepsilon = L < \sum_{i=0}^{k} a_i < U = A + \varepsilon\,,$$

which amounts to

$$\left|\left(\sum_{i=0}^{k} a_i\right) - A\right| < \varepsilon\,.$$

Thus $\sum_{i=0}^{\infty} a_i$ converges to A in the sense of $(*)$.

Hence the two definitions are equivalent. ■