Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2009

Solution to Assignment #1

In solving the following problem, you may assume without further ado that for any $x > 0$ and $n \geq 0$,

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + R_n(x)\,,$$

where

$$0 < R_n(x) < \frac{3^x x^{n+1}}{(n+1)!}\,.$$

(Not to worry, we'll show this is true later in the course.)

1. Show that e is irrational. *[10]*

Hint: Suppose e were rational. Try to derive a contradiction from this assumption by rewriting e using the expression above and then playing with it ...

SOLUTION. Suppose, by way of contradiction, that e were rational, *i.e.* $e = \frac{a}{b}$ for some positive integers a and b. Note that $b \geq 1$ and pick an n such that $n > 3b$.

Using the given equation,

$$\begin{aligned}\frac{a}{b} = e = e^1 &= 1 + \frac{1}{1!} + \frac{1^2}{2!} + \frac{1^3}{3!} + \cdots + \frac{1^n}{n!} + R_n(1) \\ &= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + R_n(1)\,.\end{aligned}$$

Multiplying through by $n!$ gives us the following equation:

$$\frac{n!a}{b} = n! + \frac{n!}{1!} + \frac{n!}{2!} + \frac{n!}{3!} + \cdots + \frac{n!}{n!} + n!R_n(1)$$

Note that since $n > 3b > b$, b is a factor of $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, and so $\frac{n!a}{b}$ must be an integer. It is easy to see that $n!$, $\frac{n!}{1!}$, $\frac{n!}{2!}$, ... , $\frac{n!}{n!}$ must all be integers too. It follows that $n!R_n(1)$ must also be an integer.

On the other hand, we have that

$$0 < R_n(1) < \frac{3^1 1^{n+1}}{(n+1)!} = \frac{3}{(n+1)!}\,,$$

so

$$0 = n!0 < n!R_n(1) < \frac{n!3}{(n+1)!} = \frac{3}{n+1}\,.$$

Since $n > 3b \geq 3$, $n+1 > 4$, and so

$$0 = n!0 < n!R_n(1) < \frac{3}{n+1} < \frac{3}{4} < 1\,,$$

which means $n!R_n(1)$ cannot be an integer, contradicting the conclusion reached earlier.

Since assuming otherwise leads to a contradiction, e cannot be rational, *i.e.* it is irrational. ■