Mathematics 3790H – Analysis I: Introduction to analysis

Trent University, Fall 2009

Solutions to the quizzes

Quiz #1. Thursday, 24 September, 2009 *(10 minutes)*

The series $\sum_{n=0}^{\infty} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ sums to 2. Denote the kth partial sum of this series by $S_k = \sum_{n=0}^{k} \frac{1}{2^n} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k}$.

1. Show that $S_k < 2$ for every $k \geq 0$. *[2]*

2, How large does k need to be to ensure that the partial sum $S_k = \sum_{n=0}^{k} \frac{1}{2^n}$ of this series is within 0.001 of 2? *[3]*

Hints: First, what, exactly, is $2 - S_k$? Second, note that $2^{10} = 1024$.

Solutions. The first question is over with very quickly if you remember the formula for the sum of a finite geometric series. The solution below does it in a brutally simple-minded way instead.

1. Consider the first few values of $2 - S_k = 2 - \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k}\right)$,

$$\begin{aligned}
2 - S_0 &= 2 - 1 = 0 \\
2 - S_1 &= 2 - \left(1 + \frac{1}{2}\right) = \frac{1}{2} \\
2 - S_2 &= 2 - \left(1 + \frac{1}{2} + \frac{1}{4}\right) = \frac{1}{4} \\
2 - S_3 &= 2 - \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = \frac{1}{8}\,,
\end{aligned}$$

and observe that in each case $2 - S_k = \frac{1}{2^k}$, which is the last term in S_k. It isn't too hard to check that this is true in general. For example,

$$\begin{aligned}
S_k + \frac{1}{2^k} &= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^k}\right) + \frac{1}{2^k} \\
&= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}}\right) + \left(\frac{1}{2^k} + \frac{1}{2^k}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-1}}\right) + \frac{1}{2^{k-1}} \\
&= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-2}}\right) + \left(\frac{1}{2^{k-1}} + \frac{1}{2^{k-1}}\right) \\
&= \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^{k-2}}\right) + \frac{1}{2^{k-2}} \\
&\;\;\vdots \\
&= \left(1 + \frac{1}{2} + \frac{1}{4}\right) + \frac{1}{4} = \left(1 + \frac{1}{2}\right) + \frac{1}{2} = 1 + 1 = 2\,.
\end{aligned}$$

Since $2 - S_k = \frac{1}{2^k} > 0$ for every $k \geq 0$, we must have $S_k < 2$ for every $k \geq 0$. ■

2. We need to find out for which values of k we have $2 - S_k < 0.001 = \frac{1}{1000}$. From our work in question 1, we know that $2 - S_k = \frac{1}{2^k}$, so we are looking for the ks such that $\frac{1}{2^k} < \frac{1}{1000}$, *i.e.* such that $2^k > 1000$. Since 2^k is an increasing function of k, $2^9 = 512 < 1000$, and $2^{10} = 1024 > 1000$, it follows that $2 - S_k < 0.001$ for all $k \geq 10$, but not for $0 \leq k \leq 9$.

   Thus k needs to be at least 10 to ensure that $2 - S_k < 0.001$. ■

Quiz #2. Thursday, 1 October, 2009 *(10 minutes)*

You may assume that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$ converges to $\frac{1}{1-x}$ for $|x| < 1$. Find the sum of each of the following series for $|x| < 1$:

1. $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$ *[2]*

2. $\sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots$ *[3]*

Hints: Substitution. Calculus.

Solutions. We will obtain both sums by modifying the given geometric series using substitution and/or calculus, after a little bit of reverse-engineering on the series in questions 1 and 2.

1. Note that

$$\begin{aligned}
\frac{d}{dx} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} &= \frac{d}{dx}\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) \\
&= \frac{d}{dx}x - \frac{d}{dx}\frac{x^3}{3} + \frac{d}{dx}\frac{x^5}{5} - \frac{d}{dx}\frac{x^7}{7} + \cdots \\
&= 1 - x^2 + x^4 - x^6 + \cdots
\end{aligned}$$

This is a geometric series with initial term 1 and common ratio $-x^2$, which therefore sums to $\frac{1}{1-(-x^2)} = \frac{1}{1+x^2}$. It follows that, up to a constant C,

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \int \left(1 - x^2 + x^4 - x^6 + \cdots\right) dx = \int \frac{1}{1+x^2}\, dx = C + \arctan(x)\,.$$

Since $\arctan(0) = 0 = \sum_{n=0}^{\infty} \frac{(-1)^n 0^{2n+1}}{2n+1}$, the constant of integration turns out to be 0, and so

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan(x)\,. \qquad \blacksquare$$

2. Note that, up to a constant C,

$$\begin{aligned}
\int \left(\sum_{n=0}^{\infty} (n+1)x^n\right) dx &= \int \left(1 + 2x + 3x^2 + 4x^3 + \cdots\right) dx \\
&= \int 1\, dx + \int 2x\, dx + \int 3x^2\, dx + \int 4x^3\, dx + \cdots \\
&= C + x + x^2 + x^3 + x^4 + \cdots
\end{aligned}$$

We could optimistically assume that $C = 1$, making the sum of the last series $\frac{1}{1-x}$, and get away with it because C will disappear in what follows:

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+1)x^n &= \frac{d}{dx}\left(C + x + x^2 + x^3 + x^4 + \cdots\right) \\
&= \frac{d}{dx}\left(1 + x + x^2 + x^3 + x^4 + \cdots\right) \qquad \text{(Since } \tfrac{d}{dx}C = 0 = \tfrac{d}{dx}1.\text{)} \\
&= \frac{d}{dx}\left(\frac{1}{1-x}\right) \\
&= \frac{-1}{(1-x)^2} \cdot \frac{d}{dx}(1-x) \\
&= \frac{-1}{(1-x)^2} \cdot (-1) \\
&= \frac{1}{(1-x)^2} \qquad \blacksquare
\end{aligned}$$

Quiz #3. Thursday, 8 October, 2009 *(10 minutes)*

1. Show that the sequence $y_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} - \ln(n)$ is decreasing. *[5]*

SOLUTION. We will show that $y_{n+1} < y_n$ by considering $y_{n+1} - y_n$:

$$\begin{aligned}
y_{n+1} - y_n &= \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n+1} - \ln(n+1)\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n)\right) \\
&= \frac{1}{n+1} - \ln(n+1) + \ln(n) = \frac{1}{n+1} - \ln\left(\frac{n+1}{n}\right) = \frac{1}{n+1} - \ln\left(1 + \frac{1}{n}\right) \\
&= \frac{1}{n+1} - \left(\frac{1}{n} - \frac{1}{2}\left(\frac{1}{n}\right)^2 + \frac{1}{3}\left(\frac{1}{n}\right)^3 - \frac{1}{4}\left(\frac{1}{n}\right)^4 + \cdots\right) \\
&= \frac{1}{n+1} - \frac{1}{n} + \frac{1}{2}\left(\frac{1}{n}\right)^2 - \frac{1}{3}\left(\frac{1}{n}\right)^3 + \frac{1}{4}\left(\frac{1}{n}\right)^4 - \cdots \\
&= \left(\frac{1}{n+1} - \frac{1}{n} + \frac{1}{2n^2}\right) + \left(-\frac{1}{3n^3} + \frac{1}{4n^4}\right) + \left(-\frac{1}{5n^5} + \frac{1}{6n^6}\right) + \cdots
\end{aligned}$$

Note that all the groupings after the first in this sum are negative, since a larger (absolute) value for the denominator means a smaller (absolute) value for the fraction. The first grouping will be non-positive for $n \geq 1$ (which are the ns for which the definition of y_n makes sense) because

$$\frac{1}{n+1} - \frac{1}{n} + \frac{1}{2n^2} = \frac{2n^2 - 2n(n+1) + (n+1)}{2n^2(n+1)} = \frac{1-n}{2n^2(n+1)},$$

which has a positive denominator and a numerator which is ≤ 0 when $n \geq 1$.

It follows that $y_{n+1} - y_n < 0$, *i.e.* $y_{n+1} < y_n$, when $n \geq 1$. ■

Quiz #4. ~~Thursday, 15~~ Monday, 19 October, 2009 *(10 minutes)*

Do *one* of questions 1 and 2.

1. Use Lagrange's Remainder Theorem to determine the number of terms of the of the partial sum for the power series expansion of $f(x) = \ln(1+x)$ that are needed to guarantee that the partial sum is within 0.1 of $\ln(2) = \ln(1+1)$. *[5]*

   Hint: You may assume that the power series expansion of $f(x)$ is $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^n}{n} + \cdots$ and that $f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{(1+x)^n}$ for $n \geq 1$.

2. Use the Intermediate Value Theorem to show that every real number $\alpha > 0$ has a square root. *[5]*

   Hint: α has a square root if $f(x) = x^2$ takes on the value α ...

SOLUTION TO 1. Recall from class or the text that

$$f(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^n x^n}{n} + R_n(x)\,,$$

where, by Lagrange's Remainder Theorem, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}$ for some c between 0 and x.

For $\ln(2) = \ln(1+1)$ we have $x = 1$, and $f^{(n+1)}(c) = \frac{(-1)^{n+2}n!}{(1+c)^{n+1}}$. This lets us estimate $|R_n(1)|$ using the Lagrange Remainder Theorem with some c such that $0 < c < 1$:

$$\begin{aligned}
|R_n(1)| &= \left|\frac{f^{(n+1)}(c)}{(n+1)!}1^{n+1}\right| = \left|\frac{(-1)^{n+2}n!}{(1+c)^{n+1}} \cdot \frac{1}{(n+1)!}\right| \\
&= \frac{1}{(1+c)^{n+1}(n+1)} \\
&< \frac{1}{(1+0)^{n+1}(n+1)} = \frac{1}{n+1}
\end{aligned}$$

We can therefore insure that $1 - \frac{1^2}{2} + \frac{1^3}{3} - \frac{1^4}{4} + \cdots + \frac{(-1)^n 1^n}{n}$ is within 0.1 of $\ln(2) = \ln(1+1)$ by ensuring that $|R_n(1)| < \frac{1}{n+1} \leq 0.1 = \frac{1}{10}$. It's pretty obvious that $\frac{1}{n+1} \leq \frac{1}{10}$ when $n + 1 \geq 10$, *i.e.* when $n \geq 9$. Taking 9 or more terms of the power series therefore ensures that the partial sum is within 0.1 of $\ln(2)$. ■

SOLUTION TO 2. Note that $f(x) = x^2$ is continuous for all x and increasing for $x \geq 0$. Suppose α is a positive real number. Choose an integer n such that $n^2 > \alpha$. $f(x) = x^2$ is a continuous function on $[0, n]$ and $f(0) = 0^2 = 0 < \alpha < n^2 = f(n)$, so, by the Intermediate Value Theorem, there is a c with $0 < c < n$ such that $c^2 = f(c) = \alpha$. Thus c is a square root of α. ■

Quiz #5. Thursday, 22 October, 2009 *(10 minutes)*

1. Suppose $f(x)$ is a function that is defined for all x near 0 and is continuous at 0, and suppose c is a real number. Use the $\varepsilon - \delta$ definition of continuity to show that $g(x) = cf(x)$ is also continuous at 0. *[5]*

SOLUTION. First, assume $c \neq 0$ and suppose that $\varepsilon > 0$. We need to find a $\delta > 0$ such that for all x, if $|x - 0| < \delta$, then $|g(x) - g(0)| < \varepsilon$. Observe that

$$\begin{aligned} |g(x) - g(0)| < \varepsilon &\iff |cf(x) - cf(0)| < \varepsilon \\ &\iff |c|\,|f(x) - f(0)| < \varepsilon \\ &\iff |f(x) - f(0)| < \frac{\varepsilon}{|c|}\,, \end{aligned}$$

where the last step requires the assumption that $c \neq 0$. Since $f(x)$ is continuous at 0, there is a $\delta > 0$ such that for all x, if $|x - 0| < \delta$, then $|f(x) - f(0)| < \frac{\varepsilon}{|c|}$. This last, however, is equivalent to $|g(x) - g(0)| < \varepsilon$. It follows that $g(x)$ is continuous if $c \neq 0$.

Second, assume $c = 0$ and suppose that $\varepsilon > 0$. Pick any $\delta > 0$ you like and suppose $|x - 0| < \delta$. Then $|g(x) - g(0)| = |0f(x) - 0f(0)| = 0\,|f(x) - f(0)| = 0 < \varepsilon$. It follows that $g(x)$ is also continuous if $c = 0$. ■

Quiz #6. Thursday, 12 November, 2009 *(10 minutes)*

1. Use the $\varepsilon - \delta$ definition of continuity to show that $g(x) = \frac{1}{3x-1}$ is continuous at 1. *[5]*

SOLUTION. We need to check that for all $\varepsilon > 0$ there is a $\delta > 0$ such that if $|x-1| < \delta$, then $|g(x) - g(1)| < \varepsilon$. Note that $g(1) = \frac{1}{3 \cdot 1 - 1} = \frac{1}{2}$. Suppose $\varepsilon > 0$ is given; we will attempt to reverse-engineer a $\delta > 0$ for this ε.

$$\begin{aligned} |g(x) - g(1)| &= \left| \frac{1}{3x-1} - \frac{1}{2} \right| = \left| \frac{2 - (3x-1)}{2(3x-1)} \right| \\ &= \left| \frac{3-3x}{6x-2} \right| = \left| \frac{-3(x-1)}{6(x-1)+4} \right| \end{aligned}$$

If we require that $|x-1| < \frac{1}{2}$, *i.e.* that $\delta \leq \frac{1}{2}$, then the denominator of the last expression is bounded away from 0, $1 = -3 + 4 \leq 6(x-1) + 4 \leq 3 + 4 = 7$. This, in turn, means that

$$\frac{3|x-1|}{7} \leq \left| \frac{-3(x-1)}{6(x-1)+4} \right| \leq \frac{3|x-1|}{1} = 3|x-1|$$

Thus, if we set $\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{3}\right)$ and require that $|x-1| < \delta$, we will have that:

$$|g(x) - g(1)| = \left| \frac{-3(x-1)}{6(x-1)+4} \right| \leq 3|x-1| < 3\delta \leq 3\frac{\varepsilon}{3} = \epsilon$$

Hence $g(x)$ is continuous at 1. ■

Take-home Quiz #7. Due on Monday, 16 November, 2009

1. Suppose $f(x)$ and g(x) are function that are defined and continuous for all x near a, and such that $g(a) \neq 0$. Use the $\varepsilon - \delta$ definition of continuity to show that $h(x) = \frac{f(x)}{g(x)}$ is also continuous at a. *[5]*

Solution. ■

Quiz #8. Thursday, 19 November, 2009 *(15 minutes)*

You may assume that the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and that the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Use the Comparison Test to determine whether or not each of the following series converges.

1. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ *[1.5]* 2. $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$ *[1.5]* 3. $\sum_{n=0}^{\infty} \frac{n}{n^3+1}$ *[2]*

Solution to 1. $\sqrt{n} \leq n$ for all $n \geq 1$, so $0 \leq \frac{1}{n} \leq \frac{1}{\sqrt{n}}$ for all $n \geq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, it follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ also diverges. ■

Solution to 2. For all $n \geq 1$, $0 \leq \frac{\sin^2(n)}{n^2} \leq \frac{1}{n^2}$ because $0 \leq \sin^2(n) \leq 1$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it follows by the Comparison Test that $\sum_{n=1}^{\infty} \frac{\sin^2(n)}{n^2}$ also converges. ■

Solution to 3. For all $n \geq 1$, $n^3 \leq n^3 + 1$, so $\frac{1}{n^3+1} \leq \frac{1}{n^3}$. It follows that for all $n \geq 1$, $0 \leq \frac{n}{n^3+1} \leq \frac{n}{n^3} = \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, it then follows by the Comparison Test that $\sum_{n=0}^{\infty} \frac{n}{n^3+1}$ also converges. ■

Quiz #9. Thursday, 26 November, 2009 *(12 minutes)*

1. Use the (limit) ratio test to verify that $\sum_{n=0}^{\infty} \frac{\pi^n}{n!}$ converges absolutely. *[2]*

2. Use the convergence test(s) of your choice to determine whether $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges absolutely, converges conditionally, or diverges. *[3]*

SOLUTION TO 1. Here goes:

$$\lim_{n\to\infty} \frac{\frac{\pi^{n+1}}{(n+1)!}}{\frac{\pi^n}{n!}} = \lim_{n\to\infty} \frac{\pi^{n+1}}{(n+1)!} \cdot \frac{n!}{\pi^n} = \lim_{n\to\infty} \frac{\pi}{n+1} = 0 < 1$$

It follows by the ratio Test that the series $\sum_{n=0}^{\infty} \frac{\pi^n}{n!}$ converges absolutely. ■

SOLUTION TO 2. $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ is obviously an alternating series. ($\ln(n) > 0$ for $n \geq 2$ and $(-1)^n$ alternates sign, just in case it *wasn't* obvious ...) The absolute values of the terms of the series is decreasing: since $\ln(n+1) > \ln(n)$ for all n,

$$\left| \frac{(-1)^{n+1}}{\ln(n+1)} \right| = \frac{1}{\ln(n+1)} < \frac{1}{\ln(n)} = \left| \frac{(-1)^n}{\ln(n)} \right| .$$

Moreover, it survives the Divergence Test: Since

$$\lim_{n\to\infty} \left| \frac{(-1)^n}{\ln(n)} \right| = \lim_{n\to\infty} \frac{1}{\ln(n)} = 0$$

because $\lim_{n\to\infty} \ln(n) = \infty$, we have $\lim_{n\to\infty} \frac{(-1)^n}{\ln(n)} = 0$ too. It follows that $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ converges by the Alternating Series Test.

It remains to determine whether the series converges absolutely or conditionally. The corresponding series of positive terms, $\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$, diverges by comparison with the $\sum_{n=2}^{\infty} \frac{1}{n}$, which diverges (why?), because $0 < \frac{1}{n} < \frac{1}{\ln(n)}$ since $\ln(n) \leq n$ for all $n \geq 2$. This means that $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}$ does not converge absolutely, so it must only converge conditionally (since it does, after all, converge). ■

Quiz #10. Thursday, 3 December, 2009 *(10 minutes)*

1. Find the interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{2^n x^n}{n+1}$. *[5]*

SOLUTION. First, we find the radius of convergence using the (Limit) Ratio Test.

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{2^{n+1}x^{n+1}}{n+2}}{\frac{2^n x^n}{n+1}} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}}{n+2} \cdot \frac{n+1}{2^n x^n} \right| = \lim_{n\to\infty} \left| \frac{2^{n+1}x^{n+1}}{2^n x^n} \cdot \frac{n+1}{n+2} \right| \\
&= \lim_{n\to\infty} 2|x| \cdot \frac{n+1}{n+2} = 2|x| \cdot \lim_{n\to\infty} \frac{n+1}{n+2} = 2|x| \cdot \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} = 2|x| \cdot \frac{1+0}{1+0} \\
&= 2|x|
\end{aligned}$$

It follows by the (Limit) Ratio Test that the given series converges absolutely when $2|x| < 1$, *i.e.* when $|x| < \frac{1}{2}$, and diverges when $2|x| > 1$, *i.e.* when $|x| > \frac{1}{2}$. Hence the radius of convergence of the series is $R = \frac{1}{2}$.

It remains to determine what happens at the endpoints of the interval of convergence, *i.e.* when $x = \pm R = \pm\frac{1}{2}$. When we plug in $x = \frac{1}{2}$, we get the series

$$\sum_{n=0}^{\infty} \frac{2^n \left(\frac{1}{2}\right)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots .$$

This is just the harmonic series, which we know diverges by the p-Test. On the other hand, when we plug in $x = -\frac{1}{2}$, we get the series

$$\sum_{n=0}^{\infty} \frac{2^n \left(-\frac{1}{2}\right)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n+1} = -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots .$$

This is the negative of the alternating harmonic series, which we know converges by the Alternating Series Test.

Hence the interval of convergence of the given series is $\left[-\frac{1}{2}, \frac{1}{2}\right)$. ■

Quiz #11. Thursday, 11 December, 2009 *(10 minutes)*

1. Show that the functions $f_n(x) = 1 + x^n$ converge uniformly to $f(x) = 1$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. *[5]*

SOLUTION. We need to show that for any $\varepsilon > 0$, there is an N such that for all $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, $|f_n(x) - f(x)| < \varepsilon$.

Note first that for any $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$, we have $|x| \leq \frac{1}{2}$. It follows that

$$|f_n(x) - f(x)| = |1 + x^n - 1| = |x^n| = |x|^n \leq \left(\frac{1}{2}\right)^n = \frac{1}{2^n}\,.$$

Now suppose that $\varepsilon > 0$ is given. Choose N such that $\frac{1}{2^N} < \varepsilon$. Then, for any $n \geq N$, we have that $2^n \geq 2^N$, and so for any $x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$,

$$|f_n(x) - f(x)| \leq \frac{1}{2^n} \leq \frac{1}{2^N} < \varepsilon\,.$$

Thus the sequence of functions $f_n(x) = 1 + x^n$ converges uniformly to $f(x) = 1$ on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, as desired. ■