

**Mathematics 3790H – Analysis I: Introduction to analysis**

TRENT UNIVERSITY, Fall 2009

**Take-home Final Exam**

*Due on Tuesday, 22 December, 2009.*

**Instructions:** Do all three of parts **I** – **III**, and, if you wish, part  $\Omega$  as well. Show all the work necessary to support your conclusions. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem. However, you may not consult any other sources, nor consult or work with any other person on this exam.

**Part I.** Do problem **0** and any *two* of problems **1** – **3**.

- 0.** Suppose  $a_n$  and  $b_n$ ,  $n \geq 0$ , are sequences such that (i)  $0 \leq a_{n+1} \leq a_n$  for all  $n \geq 0$ , (ii)  $\lim_{n \rightarrow \infty} a_n = 0$ , and (iii) there is a  $B > 0$  such that for all  $N \geq 0$ ,  $\left| \sum_{n=0}^N b_n \right| \leq B$ .

Show that  $\sum_{n=0}^{\infty} a_n b_n$  converges. [10]

- 1.** Use **0** to prove the Alternating Series Test. [5]
- 2.** Show that the series  $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \dots$  converges. [5]
- 3.** Find an example of sequences  $a_n$  and  $b_n$ ,  $n \geq 0$ , which satisfy conditions (ii) and (iii) of **0**, but for which  $\sum_{n=0}^{\infty} a_n b_n$  does not converge. [5]

**Part II.** Do any *two* of **4**–**7**. [20 = 2 × 10 each]

- 4.** Use the  $\varepsilon - \delta$  definition of limits to show that  $\lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$ .
- 5.** Determine whether the series  $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)}$  converges absolutely, converges conditionally, or diverges.
- 6.** How many terms of the series  $\sum_{n=0}^{\infty} \frac{1}{2^n}$ , are needed to guarantee that the partial sum is within 0.001 of 2?
- 7.** Give an example of a series  $\sum_{n=0}^{\infty} a_n$  which diverges even though the sequence of partial sums  $S_N = \sum_{n=0}^N a_n$  has a convergent subsequence.

**Part III.** Do any *two* of problems **8** – **11**. [*20 = 2 × 10 each*]

- 8.** Find the interval of convergence of  $\sum_{n=0}^{\infty} \frac{(\frac{1}{2} - 0)(\frac{1}{2} - 1) \cdots (\frac{1}{2} - n + 1)}{n!} x^n$ . What does it sum to within its radius of convergence?
- 9.** Recall that the Taylor polynomial of degree  $n$  of  $f(x)$  at  $a$  is defined to be  $T_{n,a}(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$  and the corresponding remainder term is  $R_{n,a}(x) = f(x) - T_{n,a}(x)$ . Suppose that  $a$  is a real number and  $f(x)$  is a function such that for some particular  $n = n_0$ ,  $R_{n_0,a}(x) = \frac{f^{(n_0+1)}(a)}{(n_0+1)!} (x-a)^{n_0+1}$  for all  $x$ . Show that it follows that the Taylor series of  $f(x)$  at  $a$  converges uniformly to  $f(x)$  for all  $x$ .
- 10.** Give an example of two power series  $\sum_{n=0}^{\infty} a_n x^n$  and  $\sum_{n=0}^{\infty} b_n x^n$  which both converge for all  $x$  and such that  $\left(\sum_{n=0}^{\infty} a_n x^n\right) \left(\sum_{n=0}^{\infty} b_n x^n\right) = 1$  for all  $x$ , or show that such a pair of series cannot exist.
- 11.** Suppose that  $(a, b)$  is an interval and that for each  $n \geq 0$ ,  $f_n(x)$  is a function on  $(a, b)$  and  $a_n$  is a real number such that  $|f_n(x)| \leq a_n$  for all  $x$  in  $(a, b)$ . Show that if the series  $\sum_{n=0}^{\infty} a_n$  converges, then  $\sum_{n=0}^{\infty} f_n(x)$  converges uniformly on  $(a, b)$ .

**Part  $\Omega$ .** *This problem on the test goes round and round!*

$4\pi$ . Write a poem about analysis or mathematics in general. [*2*]

[*Total = 60*]

HAVE A NICE BREAK!