Also, if RW, the ordinate from R to PV, be produced to meet the curve again in r,

$$RW = rW,$$

and the same proof shows that

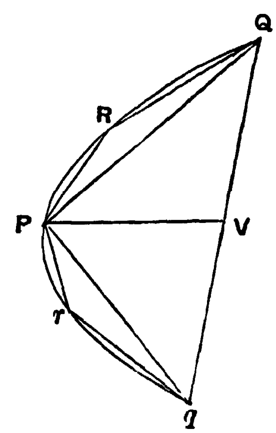
$$\triangle PQq = 8 \triangle Prq.$$

Proposition 22.

If there be a series of areas A, B, C, D, ... each of which is four times the next in order, and if the largest, A, be equal to the triangle PQq inscribed in a parabolic segment PQq and having the same base with it and equal height, then

$$(A + B + C + D + \ldots) < (\textit{area of segment } PQq).$$

For, since $\triangle PQq = 8 \triangle PRQ = 8 \triangle Pqr$, where R, r are the vertices of the segments cut off by PQ, Pq, as in the last proposition,



$$\triangle PQq = 4 (\triangle PQR + \triangle Pqr).$$

Therefore, since $\triangle PQq = A$,

$$\triangle PQR + \triangle Pqr = B.$$

In like manner we prove that the triangles similarly inscribed in the remaining segments are together equal to the area C, and so on.

Therefore $A + B + C + D + \ldots$ is equal to the area of a certain inscribed polygon, and is therefore less than the area of the segment.

Proposition 23.

Given a series of areas A, B, C, D, ... Z, of which A is the greatest, and each is equal to four times the next in order, then

$$A + B + C + \ldots + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$$

Take areas $b, c, d, \ldots$ such that

$$b = \tfrac{1}{3}B,$$

$$c = \tfrac{1}{3}C,$$

$$d = \tfrac{1}{3}D, \text{ and so on.}$$

Then, since $$b = \tfrac{1}{3}B,$$

and $$B = \tfrac{1}{4}A,$$

$$B + b = \tfrac{1}{3}A.$$

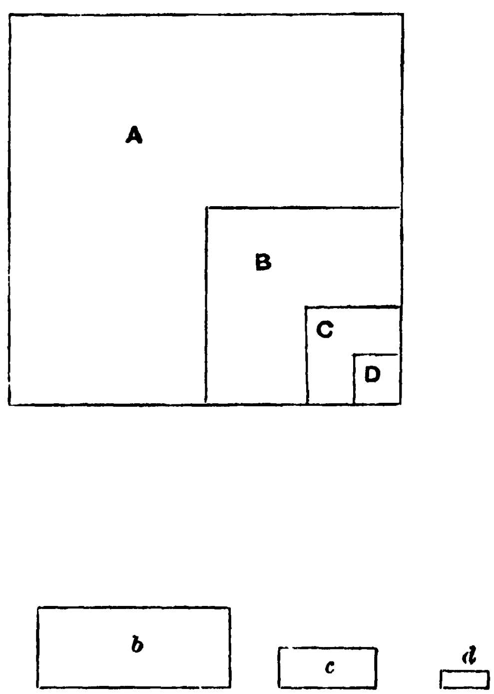
Similarly $$C + c = \tfrac{1}{3}B.$$

....

Therefore

$$B + C + D + \ldots + Z + b + c + d + \ldots + z = \tfrac{1}{3}(A + B + C + \ldots + Y).$$

But $$b + c + d + \ldots + y = \tfrac{1}{3}(B + C + D + \ldots + Y).$$



Therefore, by subtraction,

$$B + C + D + \ldots + Z + z = \tfrac{1}{3}A$$

or $$A + B + C + \ldots + Z + \tfrac{1}{3}Z = \tfrac{4}{3}A.$$

[The algebraical equivalent of this result is of course

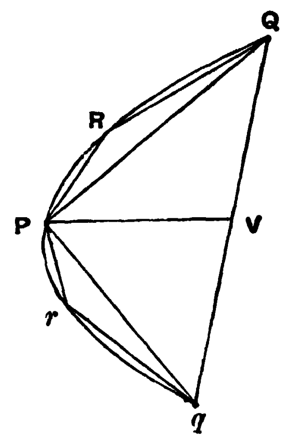
$$1+\tfrac{1}{4}+(\tfrac{1}{4})^2+\ldots+(\tfrac{1}{4})^{n-1}=\tfrac{4}{3}-\tfrac{1}{3}(\tfrac{1}{4})^{n-1}$$
$$=\frac{1-(\frac{1}{4})^n}{1-\frac{1}{4}}.\Big]$$

Proposition 24.

Every segment bounded by a parabola and a chord Qq is equal to four-thirds of the triangle which has the same base as the segment and equal height.

Suppose $$K=\tfrac{4}{3}\triangle PQq,$$

where P is the vertex of the segment; and we have then to prove that the area of the segment is equal to K.



For, if the segment be not equal to K, it must either be greater or less.

I. Suppose the area of the segment greater than K.

If then we inscribe in the segments cut off by PQ, Pq triangles which have the same base and equal height, i.e. triangles with the same vertices R, r as those of the segments, and if in the remaining segments we inscribe triangles in the same manner, and so on, we shall finally have segments remaining whose sum is less than the area by which the segment PQq exceeds K.

Therefore the polygon so formed must be greater than the area K; which is impossible, since [Prop. 23]

$$A+B+C+\ldots+Z<\tfrac{4}{3}A,$$

where $$A=\triangle PQq.$$

Thus the area of the segment cannot be greater than K.

II. Suppose, if possible, that the area of the segment is less than K.

If then $\triangle PQq = A$, $B = \frac{1}{4}A$, $C = \frac{1}{4}B$, and so on, until we arrive at an area X such that X is less than the difference between K and the segment, we have

$$A + B + C + \ldots + X + \tfrac{1}{3}X = \tfrac{4}{3}A \qquad \text{[Prop. 23]}$$
$$= K.$$

Now, since K exceeds $A + B + C + \ldots + X$ by an area less than X, and the area of the segment by an area greater than X, it follows that

$$A + B + C + \ldots + X > \text{(the segment)};$$

which is impossible, by Prop. 22 above.

Hence the segment is not less than K.

Thus, since the segment is neither greater nor less than K,

$$\text{(area of segment } PQq) = K = \tfrac{4}{3}\triangle PQq.$$