Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2009

Assignment #4
Due: Thursday, 12 November, 2009

A function from heck.

We first need a bit of notation. If x is a real number, let

$$\begin{aligned}\{x\} &= \text{the distance from } x \text{ to the nearest integer} \\ &= \min\left(x - \lfloor x \rfloor, \lceil x \rceil - x\right) .\end{aligned}$$

Note that for any real number x, $0 \le \{x\} \le \frac{1}{2}$. It will be handy later on to have a couple of basic facts about $\{x\}$ in hand.

1. For all $x \in \mathbb{R}$, $\left\{x \pm \frac{1}{2}\right\} = \frac{1}{2} - \{x\}$. *[1]*

2. For all x, $y \in \mathbb{R}$, $\{x+y\} \le \{x\} + \{y\}$ and $\{x\} - \{y\} \le \{x-y\}$. *[1]*

Here's the function from heck. For any real number x, let

$$g(x) = \sum_{n=0}^{\infty} \frac{\{n!x\}}{n!} .$$

One needs to check that this definition* really makes sense:

3. Use the Comparison Test (see Chapter 4 in the text) to verify that the series defining $g(x)$ converges no matter what x we pick. *[2]*

Note that $g(x) \ge 0$ for all $x \in \mathbb{R}$. It turns out that $g(x)$ is continuous but not differentiable at every point:

4. Show that $g(x)$ is continuous at $x = a$ for all $a \in \mathbb{R}$. *[4]*

Hint: Given an $\varepsilon > 0$, first choose an N such that $\sum\limits_{n=N+1}^{\infty} \frac{\{n!x\}}{n!} < \frac{\varepsilon}{4}$. (Note that this can be done independently of the value of x ...) Then go to work on $\sum\limits_{n=0}^{N} \frac{\{n!x\}}{n!} - \sum\limits_{n=0}^{N} \frac{\{n!a\}}{n!}$.

5. Show that $g(x)$ is not differentiable at $x = 0$. *[2]*

Hint: The idea is to construct a sequence $a_n \to 0$ such that $\left|\frac{g(a_n)-g(0)}{a_n - 0}\right| = \left|\frac{g(a_n)}{a_n}\right| \ge 1$ for all n.

Bonus Problems. *You have until Friday, 11 December, to get these in!*

6. Show that $g(x)$ is not differentiable at $x = a$ for all $a \in \mathbb{R}$. *[2]*

7. At which points x is $g(x) = 0$? *[1]*

8. At which points x is $g(x)$ rational? *[2]*

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* This function is adapted from one with similar properties given in Michael Spivak's *Calculus*.