Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #5

Recall that the harmonic series, $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, diverges, while its close relation, the alternating harmonic series, $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$, converges.

Determine, as best you can, whether each of the following relatives of the harmonic series converges or diverges.

1. $1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{\alpha(n)}{n}$, where $\alpha : \mathbb{N}^+ \to \{-1, 1\}$ is given by $\alpha(n) = \begin{cases} +1 & n = 1 \text{ or } 2 \pmod 4 \\ -1 & n = 3 \text{ or } 4 \pmod 4 \end{cases}$. *[3]*

Solution. This series converges. One way to see this is to rewrite it by grouping consecutive terms carefully:

$$\begin{aligned}
&1 + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \frac{1}{9} + \cdots \\
&= 1 + \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{5}\right) + \left(\frac{1}{6} - \frac{1}{7}\right) - \cdots + (-1)^{k+1}\left(\frac{1}{2k} - \frac{1}{2k+1}\right) + \cdots \\
&= 1 + \frac{1}{2 \cdot 3} - \frac{1}{4 \cdot 5} + \frac{1}{6 \cdot 7} - \cdots + \frac{(-1)^{k+1}}{2k(2k+1)} + \cdots \\
&= 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{4k^2 + 2k}
\end{aligned}$$

Since $\left|\frac{(-1)^{k+1}}{4k^2+2k}\right| \leq \frac{1}{4k^2}$, the rewritten series converges absolutely by comparison with the series $\sum_{k=1}^{\infty} \frac{1}{4k^2} = \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2}$. This means that the original series converges, but it can only do so *conditionally* because the underlying series of positive terms, the harmonic series, diverges. ■

2. $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} \frac{\beta(n)}{n}$, where $\beta : \mathbb{N}^+ \to \{-1, 1\}$ is given by $\beta(n) = \begin{cases} +1 & n = 1 \pmod 3 \\ -1 & n \neq 1 \pmod 3 \end{cases}$. *[3]*

Solution. This series diverges. One way to see this is to rewrite it by grouping consecutive terms carefully:

$$\begin{aligned}
&1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} - \frac{1}{9} + \cdots \\
&= \left(1 - \frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{4} - \frac{1}{5} - \frac{1}{6}\right) + \cdots + \left(\frac{1}{3k+1} - \frac{1}{3k+2} - \frac{1}{3k+3}\right) + \cdots \\
&= \frac{1}{6} - \frac{7}{60} - \frac{45}{504} - \cdots - \frac{1}{3} \cdot \frac{9k^2+6k-1}{9k^3+18k^2+11k+2} - \cdots \\
&= -\frac{1}{3} \sum_{k=0}^{\infty} \frac{9k^2+6k-1}{9k^3+18k^2+11k+2}
\end{aligned}$$

Once $k \geq 1$, $\frac{9k^2+6k-1}{9k^3+18k^2+11k+2} \geq \frac{9k^2}{9k^3+9k^3+9k^3+9k^3} = \frac{1}{4k}$, so the given series must diverge by comparison with the series $\sum\limits_{k=1}^{\infty} \frac{1}{4k} = \frac{1}{4} \sum\limits_{k=1}^{\infty} \frac{1}{k}$. ■

3. $1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \frac{1}{13} + \cdots = \sum\limits_{n=1}^{\infty} \frac{\gamma(n)}{n}$, where $\gamma : \mathbb{N}^+ \to \{-1, 1\}$ delivers a block $+1$s of length k followed by a block of -1s of length k for $k = 1, 2, 3, \ldots$. *[3]*

Solution. We will use the Alternating Series Test to show that the given series converges. (Of course, it also can only do so conditionally.) In order to do this we will rewrite the series by combining blocks of terms from the series:

$$\begin{aligned}
&1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \frac{1}{13} + \cdots \\
&= (1) - \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) - \left(\frac{1}{5} + \frac{1}{6}\right) + \left(\frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right) - \left(\frac{1}{10} + \frac{1}{11} + \frac{1}{12}\right) + \cdots \\
&= B_1^+ - B_1^- + B_2^+ - B_2^- + B_3^+ - B_3^- + \cdots + B_k^+ - B_k^- + \cdots
\end{aligned}$$

Here B_k^+ denotes the sum of the terms in the block with k +s and B_K^- denotes the sum the absolute values of the terms in the immediately following block with k −s. We need to work out which terms of the underlying harmonic series end up in each of B_k^+ and B_k^-.

The key here is to work out how many terms of $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ occur before the first term that goes into B_k^+. Since B_k^+ sums the terms that go with the block of k +s, we have $1+1+2+2+3+3+\cdots+(k-1)+(k-1)$ terms occur before the first term that goes into B_k^+. Since $1+2+3+\cdots+(k-1) = k(k-1)/2$, there are $k(k-1) = k^2 - k$ terms before the first that makes it into B_k^+. It follows that

$$\begin{aligned}
B_k^+ &= \frac{1}{k^2-k+1} + \frac{1}{k^2-k+2} + \cdots + \frac{1}{k^2-k+k} \\
&= \frac{1}{k^2-k+1} + \frac{1}{k^2-k+2} + \cdots + \frac{1}{k^2}
\end{aligned}$$

and

$$B_k^- = \frac{1}{k^2+1} + \frac{1}{k^2+2} + \cdots + \frac{1}{k^2+k}\,.$$

As noted previously, we will use the Alternating Series Test to show that

$$B_1^+ - B_1^- + B_2^+ - B_2^- + B_3^+ - B_3^- + \cdots + B_k^+ - B_k^- + \cdots = \sum_{k=1}^{\infty} \left(B_k^+ - B_K^-\right)$$

converges. First, this is clearly an alternating series.

Second, we need to show that $B_1^+ \geq B_1^- \geq B_2^+ \geq B_2^- \geq \cdots \geq B_k^+ \geq B_k^- \geq B_{k+1}^+ \geq \dots$ Since each of the k terms that goes into B_k^+ is larger than each of the k terms that goes into B_k^-, we must have that $B_k^+ \geq B_k^-$ for each k. To see that each $-$ block is larger than the next $+$ block, *i.e.* $B_k^- \geq B_{k+1}^+$, observe that

$$B_k^- = \frac{1}{k^2+1} + \frac{1}{k^2+2} + \cdots + \frac{1}{k^2+k} \geq \frac{1}{k^2+k} + \frac{1}{k^2+k} + \cdots + \frac{1}{k^2+k} = \frac{k}{k^2+k} = \frac{1}{k+1}$$

(since $\frac{1}{k^2+k}$ is the smallest of the terms involved) and that

$$\begin{aligned} B_{k+1}^+ &= \frac{1}{(k+1)^2-(k+1)+1} + \frac{1}{(k+1)^2-(k+1)+2} + \cdots + \frac{1}{(k+1)^2-(k+1)+k} \\ &= \frac{1}{k^2+k+1} + \frac{1}{k^2+k+2} + \cdots + \frac{1}{k^2+k+(k+1)} \leq \frac{k+1}{k^2+k+1} \end{aligned}$$

(since $\frac{1}{k^2+k+1}$ is the largest of the terms involved). As $(k+1)(k+1) = k^2+2k+1 \geq k^2 + k + 1 = 1 \cdot \left(k^2+k+1\right)$, a little cross-multiplication tells us that $\frac{k+1}{k^2+k+1} \leq \frac{1}{k+1}$. It follows that $B_k^- \geq B_{k+1}^+$, and hence that $B_1^+ \geq B_1^- \geq B_2^+ \geq B_2^- \geq \cdots \geq B_k^+ \geq B_k^- \geq B_{k+1}^+ \geq \dots$, as required.

Third, we need to check that the series passes the Divergence Test. From the above we know that $0 \leq B_{k+1}^- \leq B_{k+1}^+ \leq \frac{k+1}{k^2+k+1}$. Since $\lim\limits_{k\to\infty} \frac{k+1}{k^2+k+1} = 0$ (check this for yourselves!), it follows that $\lim\limits_{k\to\infty} B_{k+1}^+ = \lim\limits_{k\to\infty} B_{k+1}^- = 0$.

Hence, by the Alternating Series Test,

$$\begin{aligned} &1 - \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} - \frac{1}{10} - \frac{1}{11} - \frac{1}{12} + \frac{1}{13} + \cdots \\ &= B_1^+ - B_1^- + B_2^+ - B_2^- + B_3^+ - B_3^- + \cdots + B_k^+ - B_k^- + \cdots \end{aligned}$$

converges. ■

4. $\sum\limits_{n=1}^{\infty} \frac{\tau(n)}{n}$, where $\tau : \mathbb{N}^+ \to \{-1, 1\}$ randomly chooses, with equal probability, one of $+1$ and -1 for each $n > 0$. *[1]*

Solution. The best you can hope for with this question is to get an answer along the lines of: *"The probability that such a series $\sum\limits_{n=1}^{\infty} \frac{\tau(n)}{n}$ converges is _."* How to compute, or estimate, that probability? Well, ummm, ah, ... ■