Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #4
Math Trek: Dilithium? No, dilogarithm!

The *dilogarithm* function, $\mathrm{Li}_2(x)$, is usually defined as the sum of an infinite series:

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \dots$$

To answer the questions below you will probably want to review the basic information on convergence of series from your first-year calculus text, especially the (simplest forms of the) Comparison Test and the Integral Test.

1. Show that the series defining $\mathrm{Li}_2(x)$ converges for all x with $-1 \leq x \leq 1$. *[3]*

Solution. If $-1 \leq x \leq 1$, then $\left|\frac{x^n}{n^2}\right| \leq \frac{1}{n^2}$. Since the series $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is known to converge (to $\frac{\pi^2}{6}$; see Assignment #1), it follows that $\sum\limits_{n=1}^{\infty} \frac{x^n}{n^2}$ converges absolutely (and hence converges) for $-1 \leq x \leq 1$. ■

2. How is the dilogarithm function related to the natural logarithm function? *[3]*

Solution. Recall that

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n} = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots\right),$$

which is a lot like the series defining $\mathrm{Li}_2(x)$,

$$\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = x + \frac{x^2}{4} + \frac{x^3}{9} + \frac{x^4}{16} + \dots,$$

except for the minus sign in front and the fact that the denominators are n instead of n^2. We can make the the series for $\mathrm{Li}_2(x)$ look more like that for $\ln(1-x)$ by taking its derivative:

$$\begin{aligned}
\frac{d}{dx}\mathrm{Li}_2(x) &= \frac{d}{dx}\left(\sum_{n=1}^{\infty} \frac{x^n}{n^2}\right) = \sum_{n=1}^{\infty} \frac{d}{dx}\left(\frac{x^n}{n^2}\right) \\
&= \frac{d}{dx}x + \frac{d}{dx}\frac{x^2}{4} + \frac{d}{dx}\frac{x^3}{9} + \frac{d}{dx}\frac{x^4}{16} + \dots + \frac{d}{dx}\frac{x^n}{n^2} + \dots \\
&= 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \dots + \frac{x^{n-1}}{n} + \dots = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n}
\end{aligned}$$

This is almost the same, except for the minus sign in front and a surplus power of x in every term, which problems are easy to fix:

$$\frac{d}{dx}\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{x^n}{n} = \frac{1}{x}\left(-\ln(1-x)\right) = -\frac{1}{x}\ln(1-x)$$

For those who dislike dividing by zero*, this can be rearranged a little:

$$x\frac{d}{dx}\mathrm{Li}_2(x) = -\ln(1-x)$$

There is also, of course, a corresponding integral formula ... ■

3. Denote the kth remainder term at 0 of the dilogarithm function by:

$$R_{k,0}(x) = \mathrm{Li}_2(x) - \sum_{n=1}^{k} \frac{x^n}{n^2} = \mathrm{Li}_2(x) - \left(x + \frac{x^2}{4} + \frac{x^3}{9} + \cdots + \frac{x^k}{k^2}\right)$$

Show that for any $\varepsilon > 0$ there is an $K > 0$ such that for any $k \geq K$, $|R_{k,0}(x)| < \varepsilon$ for all x with $-1 \leq x \leq 1$. *[4]*

Solution. First, note that:

$$R_{k,0}(x) = \mathrm{Li}_2(x) - \sum_{n=1}^{k} \frac{x^n}{n^2} = \sum_{n=k+1}^{\infty} \frac{x^n}{n^2} = \frac{x^{k+1}}{(k+1)^2} + \frac{x^{k+2}}{(k+2)^2} + \dots$$

Second, reusing the observations from the solution to **1**, when $-1 \leq x \leq 1$, we get:

$$\begin{aligned} |R_{k,0}(x)| &= \left|\frac{x^{k+1}}{(k+1)^2} + \frac{x^{k+2}}{(k+2)^2} + \dots\right| \leq \left|\frac{x^{k+1}}{(k+1)^2}\right| + \left|\frac{x^{k+2}}{(k+2)^2}\right| + \dots \\ &\leq \frac{1}{(k+1)^2} + \frac{1}{(k+2)^2} + \cdots = \sum_{n=k+1}^{\infty} \frac{1}{n^2} \end{aligned}$$

Third, recall that $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is a series of positive terms converging to $\frac{\pi^2}{6}$. If $S_k = \sum\limits_{n=1}^{k} \frac{1}{n^2}$ is the kth partial sum, it follows that $S_1 < S_2 < S_3 < \cdots < \frac{\pi^2}{6}$ and that $\lim\limits_{k\to\infty} S_k = \frac{\pi^2}{6}$. This means that given an $\varepsilon > 0$, there is a $K > 0$ such that for any $k \geq K$, $\left|S_k - \frac{\pi^2}{6}\right| = \frac{\pi^2}{6} - S_k < \varepsilon$. Finally, it now follows that:

$$|R_{k,0}(x)| \leq \sum_{n=k+1}^{\infty} \frac{1}{n^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - \left(\sum_{n=1}^{k} \frac{1}{n^2}\right) = \frac{\pi^2}{6} - S_k < \varepsilon \qquad ■$$

* Friends don't let friends divide by zero!