Mathematics 3790H – Analysis I: Introduction to analysis

Trent University, Fall 2008

Solutions to Assignment #2
The integral form of the remainder of a Taylor series

In what follows, let us suppose that a is a real number and $f(x)$ is a function such that $f^{(n)}(x)$ is defined and continuous for all $n \geq 0$ and all values of x we may encounter. Recall that for $n \geq 0$, the Taylor polynomial of degree n of $f(x)$ at a is

$$\begin{aligned} T_{n,a}(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \\ &= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n \,, \end{aligned}$$

and that the corresponding remainder term is

$$R_{n,a}(x) = f(x) - T_{n,a}(x) \,.$$

1. Use the Fundamental Theorem of Calculus to show that

$$R_{0,a}(x) = \int_a^x f'(t)\,dt \,. \quad \textit{[1]}$$

Solution.

$$\begin{aligned} \int_a^x f'(t)\,dt &= f(x) - f(a) && \text{(By the Fundamental Theorem of Calculus.)} \\ &= f(x) - T_{0,a}(x) && \text{(By the definition of } T_{0,a}(x).\text{)} \\ &= R_{0,a}(x) && \text{(By the definition of } R_{0,a}(x).\text{)} \ \blacksquare \end{aligned}$$

2. Use the formula in **1** and integration by parts to show that

$$R_{1,a}(x) = \int_a^x f''(t)(x-t)\,dt \,. \quad \textit{[2]}$$

Hint: Use the parts $u = f'(t)$ and $v = t - x$ ...

Solution. We start with the integral in **1** and apply parts with $u = f'(t)$ and $v = t - x$, so $du = f''(t)$ and $dv = dt$.

$$\begin{aligned} \int_a^x f'(t)\,dt &= f'(t)(t-x)\Big|_a^x - \int_a^x f''(t)(t-x)\,dt \\ &= f'(x)(x-x) - f'(a)(a-x) + \int_a^x f''(t)(x-t)\,dt \\ &= f'(a)(x-a) + \int_a^x f''(t)(x-t)\,dt \end{aligned}$$

It follows that

$$\begin{aligned}
\int_a^x f''(t)(x-t)\,dt &= \int_a^x f'(t)\,dt - f'(a)(x-a) \\
&= R_{0,a}(x) - f'(a)(x-a) \qquad \text{(By } \mathbf{1}.\text{)} \\
&= f(x) - f(a) - f'(a)(x-a) \\
&= f(x) - T_{1,a}(x) \\
&= R_{1,a}(x)\,,
\end{aligned}$$

as desired. ■

3. Use the formula in **2** and integration by parts to show that

$$R_{2,a}(x) = \int_a^x \frac{f^{(3)}(t)}{2}(x-t)^2\,dt\,. \quad \textit{[2]}$$

Solution. We start with the integral in **2** and apply parts with $u = f''(t)$ and $v = \frac{1}{2}(t-x)^2$, so $du = f^{(3)}(t)$ and $dv = (t-x)dt$.

$$\begin{aligned}
\int_a^x f''(t)(x-t)\,dt &= -\int_a^x f''(t)(t-x)\,dt \\
&= -\left(\frac{f''(t)}{2}(t-x)^2 \Big|_a^x - \int_a^x \frac{f^{(3)}(t)}{2}(t-x)^2\,dt \right) \\
&= -\left(\frac{f''(x)}{2}(x-x)^2 - \frac{f''(a)}{2}(a-x)^2 - \int_a^x \frac{f^{(3)}(t)}{2}(t-x)^2\,dt \right) \\
&= \frac{f''(a)}{2}(a-x)^2 + \int_a^x \frac{f^{(3)}(t)}{2}(t-x)^2\,dt \\
&= \frac{f''(a)}{2}(x-a)^2 + \int_a^x \frac{f^{(3)}(t)}{2}(x-t)^2\,dt
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_a^x \frac{f^{(3)}(t)}{2}(x-t)^2\,dt &= \int_a^x f''(t)(x-t)\,dt - \frac{f''(a)}{2}(x-a)^2 \\
&= R_{1,a}(x) - \frac{f''(a)}{2}(x-a)^2 \qquad \text{(By } \mathbf{2}.\text{)} \\
&= f(x) - f(a) - f'(a)(x-a) - \frac{f''(a)}{2}(x-a)^2 \\
&= f(x) - T_{2,a}(x) \\
&= R_{2,a}(x)\,,
\end{aligned}$$

as desired. ■

4. Find an integral formula for $R_{n,a}$ and use induction to show that it works. *[5]*

Solution. We will use induction on n to show that

$$R_{n,a}(x) = \int_a^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n\,dt$$

for $n \geq 0$.

Base Step. ($n = 0$) This is just **1**. (Note that $0! = 1$ and $(x-t)^0 = 1$.)

Inductive Hypothesis. ($n = k$)

$$R_{k,a}(x) = \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt$$

Inductive Step. ($n = k \to n = k+1$) We assume the inductive hypothesis and apply integration by parts to the integral, with $u = f^{(k+1)}(t)$ and $v = \dfrac{1}{(k+1)!}(t-x)^{k+1}$, so $du = f^{(k+2)}(t)dt$ and $dv = \dfrac{1}{(k+1)!}(k+1)(t-x)^k dt = \dfrac{1}{k!}(t-x)^k dt$.

$$\begin{aligned}
\int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt &= (-1)^k \int_a^x \frac{f^{(k+1)}(t)}{k!}(t-x)^k\,dt \\
&= (-1)^k \left(\frac{f^{(k+1)}(t)}{(k+1)!}(t-x)^{k+1}\bigg|_a^x - \int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(t-x)^{k+1}\,dt \right) \\
&= (-1)^k \left(\frac{f^{(k+1)}(x)}{(k+1)!}(x-x)^{k+1} - \frac{f^{(k+1)}(a)}{(k+1)!}(a-x)^{k+1} \right. \\
&\qquad\qquad \left. - \int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(t-x)^{k+1}\,dt \right) \\
&= (-1)^k \left(-\frac{f^{(k+1)}(a)}{(k+1)!}(a-x)^{k+1} - \int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(t-x)^{k+1}\,dt \right) \\
&= (-1)^{k+1} \frac{f^{(k+1)}(a)}{(k+1)!}(a-x)^{k+1} \\
&\qquad\qquad + (-1)^{k+1} \int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(t-x)^{k+1}\,dt \\
&= \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} + \int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt
\end{aligned}$$

It follows that

$$\begin{aligned}
\int_a^x \frac{f^{(k+2)}(t)}{(k+1)!}(x-t)^{k+1}\,dt &= \int_a^x \frac{f^{(k+1)}(t)}{k!}(x-t)^k\,dt - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \\
&= R_{k,a}(x) - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \\
&\qquad\qquad\qquad\qquad \text{(By the inductive hypothesis.)} \\
&= f(x) - f(a) - f'(a)(x-a) - \cdots - \frac{f^{(k)}(a)}{(k)!}(x-a)^k \\
&\qquad\qquad - \frac{f^{(k+1)}(a)}{(k+1)!}(x-a)^{k+1} \\
&= f(x) - T_{k+1,a}(x) \\
&= R_{k+1,a}(x)\,,
\end{aligned}$$

as desired. Whew! ■