Mathematics 3790H – Analysis I: Introduction to analysis
TRENT UNIVERSITY, Fall 2008

Solutions to the quizzes

Quiz #1. Wednesday, 17 September, 2008. *[10 minutes]*

1. Find the sum of the series $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$. *[2]*

Solution. The series $\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$ is a geometric series with initial term 1 and common ratio $\frac{2}{3}$. Since $\left|\frac{2}{3}\right| < 1$, the series converges to $\frac{1}{1-\frac{2}{3}} = \frac{1}{1/3} = 3$. ■

2. Verify that the geometric series $\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$ and $\sum_{n=0}^{\infty} (x-1)^n$ are equal for any x for which they both converge. *[3]*

Solution. We will use the formula for the sum of a geometric series (when it converges) twice. First,

$$\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}} = \sum_{n=0}^{\infty} \frac{1}{2}\left(\frac{x}{2}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{x}{2}\right)^n = \frac{1}{2}\cdot\frac{1}{1-\frac{x}{2}} = \frac{1}{2-x}\,.$$

when the series $\sum_{n=0}^{\infty} \frac{x^n}{2^{n+1}}$ converges. Second,

$$\sum_{n=0}^{\infty}(x-1)^n = \frac{1}{1-(x-1)} = \frac{1}{2-x}\,,$$

when the series $\sum_{n=0}^{\infty} (x-1)^n$ converges. It follows that the two series are equal for any x for which they both converge. ■

Quiz #2. Wednesday, 24 September, 2008. *[10 minutes]*

1. Use Newton's binomial series formula to find an infinite series which sums to $\frac{1}{\sqrt{2}}$. *[5]*

Solution. Recall that Newton's binomial series formula is

$$(1+x)^a = 1 + ax + \frac{a(a-1)}{2!}x^2 + \frac{a(a-1)(a-2)}{3!}x^2 + \frac{a(a-1)(a-2)(a-3)}{4!}x^4 + \cdots,$$

where the series is guaranteed to converge for all a and for all x with $|x| < 1$. The trick here is to rewrite $\frac{1}{\sqrt{2}}$ in the form $(1+x)^a$ for some x with $|x| < 1$, in order to ensure the convergence of the corresponding series.

Let $a = \frac{1}{2}$ and $x = -\frac{1}{2}$. First, observe that

$$\left(1 + \left(-\frac{1}{2}\right)\right)^{1/2} = \left(\frac{1}{2}\right)^{1/2} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}.$$

Second, since $\left|-\frac{1}{2}\right| = \frac{1}{2} < 1$, the corresponding binomial series converges. Hence

$$\begin{aligned}
\frac{1}{\sqrt{2}} &= \left(1 + \left(-\frac{1}{2}\right)\right)^{1/2} \\
&= 1 + \frac{1}{2} \cdot \left(-\frac{1}{2}\right) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} \cdot \left(-\frac{1}{2}\right)^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} \cdot \left(-\frac{1}{2}\right)^3 + \cdots \\
&= 1 - \frac{1}{2^2} + \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)}{2!} \cdot \left(-\frac{1}{2}\right)^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!} \cdot \left(-\frac{1}{2}\right)^3 + \cdots \\
&= 1 - \frac{1}{2^2} - \frac{1}{2^4 \cdot 2!} - \frac{1 \cdot 3}{2^6 \cdot 3!} - \cdots - \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2^{2n+2} \cdot (n+1)!} - \cdots,
\end{aligned}$$

which will do until a better formula comes along! ■

Quiz #3. Wednesday, 1 October, 2008. *[10 minutes]*

You may assume that $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$ is the Taylor series of $f(x) = \frac{1}{1-x}$ at 0, and let $R_{n,0}(x) = D_n(0,x) = f(x) - 1 - x - x^2 - \cdots - x^{n-1}$ denote the nth remainder at 0.

1. Find $f^{(4)}(0)$. *[2]*

Solution. We know from the formula for the Taylor series that the coefficient of x^4 in the Taylor series of $f(x)$ at 0 is $\frac{f^{(4)}(0)}{4!}$. On the other hand, since the Taylor series in question is $1 + x + x^2 + x^3 + \dots$, we must have $\frac{f^{(4)}(0)}{4!} = 1$. It folows that $f^{(4)}(0) = 4! = 24$. ■

2. What does the Langrange Remainder Theorem tell you about $R_{4,0}(x)$? *[3]*

Solution. According to the Lagrange Remainder Theorem,

$$R_{4,0}(x) = D_4(0,x) = \frac{f^{(4)}(c)}{4!}x^4$$

for some real number(s) c strictly between 0 and x. ■

Quiz #4. Wednesday, 8 October, 2008. *[10 minutes]*

1. Use the $\varepsilon - \delta$ definition of limits to verify that $\lim_{x \to 0} x\cos(x) = 0$. *[5]*

Solution. We need to check that for any $\varepsilon > 0$, there is a $\delta > 0$ such that for all x, if $0 < |x - 0| < \delta$, then $|x\cos(x) - 0| < \varepsilon$.

Given an $\varepsilon > 0$, we find suitable $\delta > 0$ by reverse engineering:

$$\begin{aligned}
|x\cos(x) - 0| < \varepsilon \quad &\iff \quad |x\cos(x)| < \varepsilon \\
&\Longleftarrow \quad |x\cos(x)| \leq |x| < \varepsilon \qquad \text{(Since } |\cos(x)| \leq 1.\text{)} \\
&\iff \quad |x - 0| < \varepsilon
\end{aligned}$$

It follows that $\delta = \varepsilon$ does the job. ■

Quiz #5. Wednesday, 29 October, 2008. *(Open book!) [10 minutes]*

1. Give an example to show that the following converse to the Mean Value Theorem is *not* true. *[5]*

   > Suppose a function $f(x)$ is defined and differentiable for all x. Then, for every $x = c$, there are a and b with $a < c < b$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

   Explain (informally!) why your example does the job. *[5]*

Solution. $f(x) - x^3$ and $c = 0$ will do the job. $f'(x) = 3x^2$, so $f'(0) = 0$. On the other hand, since $x^3 < 0$ for all $x < 0$ and $x^3 > 0$ for all $x > 0$, no matter how we pick a and b with $a < 0 < b$, we'll have $\frac{f(b)-f(a)}{b-a} > 0$. ■

Quiz #6. Wednesday, 29 October, 2008. *(Open book!) [10 minutes]*

Give an example of each of the following:

1. A function which is defined for all x and is strictly decreasing but does not satisfy the intermediate value theorem. *[1]*
2. A function which defined for all x and satisfies the intermediate value property, but is not differentiable for all x. *[2]*
3. A function which is defined for all x in $[-1, 1]$ and is continuous except at $x = 0$, satisfies the intermediate value property on $[-1, 1]$, but is unbounded on $[-1, 1]$. *[2]*

Explain (informally!) why your examples do the job.

Solutions. Here goes!

1. $f(x) = \begin{cases} 1 - x & x \leq 0 \\ -x & x > 0 \end{cases}$ does the job. It is clearly defined for all x and strictly decreasing, but does not satisfy the intermediate value property: $f(-1) = 1-(-1) = 2$ and $f(1) = -1$, but there is no c with $-1 < c < 1$ such that $f(c) = \frac{1}{2}$, even though $-1 < \frac{1}{2} < 2$. $\square$

2. $g(x) = |x|$ does the job. It is clearly defined for all x and satisfies the intermediate value property since it is continuous everywhere, but is not differentiable at $x = 0$. $\square$

3. $h(x) = \begin{cases} \frac{1}{x} \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ does the job. It is defined for all x (including all $x \in [-1, 1]$) and it is easy to see that it must be continuous everywhere except at $x = 0$. Note that it satisfies the intermediate value property on $[-1, 1]$ for the same reasons that $\sin\left(\frac{1}{x}\right)$ does. Finally, it is not hard to see that it is unbounded near $x = 0$, and hence on $[-1, 1]$. $\square$

That's all, folks! $\blacksquare$

Quiz #7. Wednesday, 5 November, 2008. *[10 minutes]*

Let $f(x) = 1 + x^2 + x^4$.

1. What is the Taylor series of $f(x)$ at $a = -1$? *[2.5]*
2. What is the Cauchy form of the remainder $R_{3,-1}(x) = f(x) - T_{3,-1}(x)$? *[2.5]*
   Note: Recall that $T_{n,a}(x)$ is the polynomial in $x - a$ consisting of the terms of degree $\leq n$ of the Taylor series of $f(x)$ at a.

Bonus. Find an explicit value for the $c \in (-1, 0)$ that appears in the Cauchy form of the remainder $R_{3,-1}(0)$. *[1]*

Solutions. Recall that the Taylor series of $f(x)$ and a is $\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$, and that by the Cauchy Remainder Theorem, the remainder $R_{n,a}(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ is equal to $\frac{f^{(n+1)}(c)}{n!}(x-c)^n(x-a)$ for some c strictly between a and x.

1. We first compute the derivatives of $f(x)$:

$$\begin{aligned}
f(x) &= 1 + x^2 + x^4 \\
f'(x) &= 2x + 4x^3 \\
f''(x) &= 2 + 12x^2 \\
f^{(3)}(x) &= 24x \\
f^{(4)}(x) &= 24 \\
f^{(k)}(x) &= 0 \quad \text{for all } k > 4
\end{aligned}$$

It follows that the Taylor series of $f(x)$ at $a = -1$ is

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{f^{(k)}(-1)}{k!}(x-(-1))^k &= f(-1) + f'(-1)(x+1) + \frac{f''(-1)}{2!}(x+1)^2 \\
&\quad + \frac{f^{(3)}}{3!}(x+1)^3 + \frac{f^{(4)}}{4!}(x+1)^4 + \frac{f^{(5)}}{5!}(x+1)^5 + \cdots \\
&= 3 + (-6)(x+1) + \frac{14}{2}(x+1)^2 \\
&\quad + \frac{-24}{6}(x+1)^3 + \frac{24}{24}(x+1)^4 + \frac{0}{120}(x+1)^5 + \cdots \\
&= 3 - 6(x+1) + 7(x+1)^2 - 4(x+1)^3 + (x+1)^4 ,
\end{aligned}$$

since $f^{(k)}(-1) = 0$ for all $k > 4$. □

2. By the Cauchy Remainder Theorem, the remainder

$$R_{3,-1}(x) = \frac{f^{(3+1)}(c)}{3!}(x-c)^3\,(x-(-1)) = \frac{24}{6}(x-c)^3(x+1) = 4(x-c)^3(x+1)\,,$$

for some c strictly between a and x. □

As for the bonus, you can give it a go yourselves! ■

Quiz #8. Wednesday, 12 November, 2008. *[15 minutes]*

1. Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}} = \frac{1}{1^2} + \frac{1}{2^4} + \frac{1}{3^2} + \frac{1}{4^4} + \frac{1}{5^2} + \frac{1}{6^4} + \cdots$ converges. *[2]*

2. Suppose that $\sum_{n=0}^{\infty} a_n$ converges absolutely, $B > 0$, and that $|b_n| \leq B$ for each $n \geq 0$. Show that $\sum_{n=0}^{\infty} a_n b_n$ converges absolutely. *[3]*

Solutions. ... starring the Comparison Test and the Cauchy Criterion!

1. Note that if $n \geq 1$, then $n^4 \geq n^2$, so $\frac{1}{n^4} \leq \frac{1}{n^2}$. It follows that $0 < \frac{1}{n^{3+(-1)^n}} \leq \frac{1}{n^2}$ for each $n \geq 1$, and so the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3+(-1)^n}}$ converges by comparison with the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$. (Recall that we know that the latter series converges, to $\frac{\pi^2}{6}$.) □

2. Since $\sum_{n=0}^{\infty} a_n$ converges absolutely, it follows by the Cauchy Criterion that for any $\varepsilon > 0$ there is an N such that for all $m > k \geq N$, $\sum_{n=k}^{m} |a_n| < \varepsilon$.

   We will use the Cauchy Criterion and the observation above to show that $\sum_{n=0}^{\infty} a_n b_n$ converges absolutely too. Suppose $\varepsilon > 0$. Choose N such that for all $m > k \geq N$, $\sum_{n=k}^{m} |a_n| < \frac{\varepsilon}{B}$. Then, for all $m > k \geq N$,

$$\sum_{n=k}^{m} |a_n b_n| \leq \sum_{n=k}^{m} |a_n| B = B \sum_{n=k}^{m} |a_n| < B \cdot \frac{\varepsilon}{B} = \varepsilon \,,$$

   so $\sum_{n=k}^{m} |a_n b_n|$ converges by the Cauchy Criterion. It follows that $\sum_{n=0}^{\infty} a_n b_n$ converges absolutely. □

That's all, folks! ■

Quiz #9. Wednesday, 19 November, 2008. *[5 minutes]*

1. Use the Ratio Test to show that $\displaystyle\sum_{n=0}^{\infty} \frac{c^n}{n!}$ converges for any $c \in \mathbb{R}$. *[5]*

Solution. Suppose $c \in \mathbb{R}$. Then

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{c^{n+1}/(n+1)!}{c^n/n!} \right| \\
&= \lim_{n\to\infty} \left| \frac{c^{n+1} \cdot n!}{c^n \cdot (n+1)!} \right| \\
&= \lim_{n\to\infty} \left| \frac{c}{n+1} \right| \\
&= \lim_{n\to\infty} \frac{|c|}{n+1} = 0\,,
\end{aligned}$$

since c is constant and $n+1 \to \infty$ as $n \to \infty$. Hence the series $\displaystyle\sum_{n=0}^{\infty} \frac{c^n}{n!}$ converges by the Ratio Test, irrespective of the value of c. ■

Quiz #10. Wednesday, 26 November, 2008. *(Open book!) [7 minutes]*

1. Determine whether the series $\displaystyle\sum_{n=0}^{\infty} \frac{(2k)!}{4^k \cdot k! \cdot k!}$ converges absolutely, converges conditionally, or diverges. *[5]*

Solution. The obvious thing to try with a series whose terms combine factorials with powers is the Ratio Test. The ratio of consecutive terms of this series is:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{(2(n+1))!}{4^{n+1} \cdot (n+1)! \cdot (n+1)!} \div \frac{(2n)!}{4^n \cdot n! \cdot n!} \\
&= \frac{(2n+2)!}{4^{n+1} \cdot (n+1)! \cdot (n+1)!} \cdot \frac{4^n \cdot n! \cdot n!}{(2n)!} \\
&= \frac{(2n+2)(2n+1)}{4(n+1)(n+1)} \\
&= \frac{4n^2+6n+2}{4n^2+8n+4} \\
&= \frac{n^2+\frac{3}{2}n+\frac{1}{2}}{n^2+2n+1}
\end{aligned}$$

Unfortunately, since

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2+\frac{3}{2}n+\frac{1}{2}}{n^2+2n+1} = \lim_{n\to\infty} \frac{1+\frac{3}{2n}+\frac{1}{2n^2}}{1+\frac{2}{n}+\frac{1}{n^2}} = \frac{1+0+0}{1+0+0} = 1\,,$$

the Ratio Test is inconclusive in this case. However, we are all set up to apply Gauss' Test to resolve the matter, because the ratio of consecutive terms is a ratio of monic polynomials of the same degree.

The key is to compare the coefficients of the next-to-highest powers of n in the numerator and denominator, which are $\frac{3}{2}$ and 2, respectively, in this case. Since $\frac{3}{2} < 2$, part 3 of Gauss' Test tells us that the series will converge if it alternates, and because $\frac{3}{2} \geq 2-1 = 1$, part 4 tells us that it cannot converge absolutely. As all the terms of the series are positive, absolute convergence is the only kind of convergence we could hope to have, and since we do not have it, the series must diverge. ■

Quiz #11. Wednesday, 3 December, 2008. *[10 minutes]*

1. Find the radius and interval of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{(n+1)^3}$. *[5]*

Solution. We will use the Ratio Test to find the radius of convergence. Since

$$\begin{aligned}
\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{((n+1)+1)^3}}{\frac{x^n}{(n+1)^3}} \right| \\
&= \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{(n+1)^3}{(n+2)^3} \right| \\
&= \lim_{n\to\infty} |x| \cdot \frac{n^3+3n^2+3n+1}{n^2+6n^2+12n+8} \\
&= |x| \cdot \lim_{n\to\infty} \frac{1+\frac{3}{n}+\frac{3}{n^2}+\frac{1}{n^3}}{1+\frac{6}{n}+\frac{12}{n^2}+\frac{8}{n^3}} \\
&= |x| \cdot \lim_{n\to\infty} \frac{1+0+0+0}{1+0+0+0} = |x|\,,
\end{aligned}$$

the Ratio Test tells us that the series converges when $|x| < 1$ and diverges when $|x| > 1$. It follows that the radius of convergence is $R = 1$.

To determine the interval of convergence, we need to check whether the series converges or not at the endpoints of the interval about 0 given by the radius of convergence, namely 1 and -1. For $x = 1$ we get the series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)^3}$, which converges by comparison with the convergent series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$ because for $n \geq 0$, $(n+1)^3 \geq (n+1)^2$, so $0 \leq \dfrac{1}{(n+1)^3} \leq \dfrac{1}{(n+1)^2}$. It follows also that the series we get for $x = -1$, $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^3}$, converges absolutely, and hence converges. Thus the interval of convergence is $[-1, 1]$. ■