Mathematics 3790H – Analysis I: Introduction to analysis

Trent University, Fall 2008

Take-home Final Exam

Due: Friday, 19 December, 2008

Instructions: Do all three of parts **A** – **C**, and, if you wish, part ∞ as well. Show all your work. You may use your textbooks and notes, as well as any handouts and returned work, from this and any other courses you have taken or are taking now. You may also ask the instructor to clarify the statement of any problem. However, you may not consult any other sources, nor consult or work with any other person on this exam.

Part A. Do any *three* of problems **1** – **4**. *[10 each]*

1. Determine the radius and interval of convergence of $2 + 6x + 12x^2 + 20x^3 + \cdots = \sum_{n=0}^{\infty} (n+1)(n+2)x^n$. What does it sum to within its radius of convergence?

2. How many terms of the Taylor series for $\cos(x)$ about 0, evaluated at $x = \frac{\pi}{2}$, are needed to guarantee that the partial sum is within 0.01 of $\cos\left(\frac{\pi}{2}\right)$?

3. Use the $\varepsilon - \delta$ definition of limits to show that $\lim_{s \to 1} \left(s^2 + s - 1\right) = 1$.

4. Determine whether $\sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{4}\right)$ converges or diverges.

Part B. Do any *two* of problems **5** – **7**. *[10 each]*

5. Show that $f(x) = \begin{cases} x^2 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is continuous at $x = 0$, but not at any other point.

6. Suppose that (a, b) is an interval and that for each $n \geq 0$, $f_n(x)$ is a function on (a, b) and a_n is a real number such that $|f_n(x)| \leq a_n$ for all x in (a, b). Show that if the series $\sum_{n=0}^{\infty} a_n$ converges, then $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on (a, b).

7. Assume that every set S of real numbers that has an upper bound has a least upper bound. Show that it follows that the nested interval principle must hold.

Part C. Do any *two* of problems **8** – **10**. *[10 each]*

8. Suppose t_n, $n \geq 1$, is a sequence such that $\lim_{n \to \infty} t_n$ exists. Show that the sequence is *Cesaro-summable*, *i.e.* $\lim_{n \to \infty} \frac{t_1 + t_2 + t_3 + \cdots + t_n}{n}$ exists.

9. Suppose that the Taylor series of a function $f(x)$ has radius of convergence $R = \infty$. Can this series converge uniformly if $f(x)$ is bounded or, respectively, unbounded on $\mathbb{R}$? In each case, either give an example to show that it can or prove that it can't.

10. Suppose that $\sum_{n=0}^{\infty} a_n$ converges absolutely. Show that $\sum_{n=0}^{\infty} a_n^2$ converges.

Part ∞.

0. Write a poem about real analysis or mathematics in general. *[2]*

[Total = 70]

I hope you enjoyed the course!