Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2008

Solutions to Assignment #5

Computing determinants using the Gauss-Jordan algorithm

Recall from Assignment #5 that the determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If $i \neq j$ and you add any multiple of the jth row of $\mathbf{A}$ to the ith row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}|$.

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant, *i.e.* $|\mathbf{A}^T| = |\mathbf{A}|$.

1. Why are rules *ii*–*iv* true for the columns of an $n \times n$ matrix as well as the rows? *[2]*

Solution. Rule v is the reason[1]. Applying the operations mentioned in rules *ii* – *iv* to the columns of $\mathbf{A}$ corresponds to applying them to the rows of $\mathbf{A}^T$. Rule v tells us that $|\mathbf{B}| = |\mathbf{B}^T|$ for any matrix $\mathbf{B}$, so the effect on $|\mathbf{A}|$ of column operations on $\mathbf{A}$ is exactly the same as the effect on $|\mathbf{A}^T|$ of the corresponding row operations on $\mathbf{A}^T$. Hence rules *ii* – *iv* work for columns as well as rows. ■

2. Use rules *i*–*v* and **1** to find the determinant of an $n \times n$ matrix $\mathbf{A}$ if:

a. $\mathbf{A}$ has a column or a row of zeros. *[2]*

Solution. Suppose $\mathbf{A}$ is an $n \times n$ matrix whose ith row, call it $\mathbf{r}_i$, is all zeros. Note that in this case $\mathbf{r}_i = 0\mathbf{r}_i$, so, by rule *iii*, $|\mathbf{A}| = 0|\mathbf{A}| = 0$.

If $\mathbf{A}$ has a column of zeros instead, then $\mathbf{A}^T$ must have a row of zeros, so $|\mathbf{A}| = |\mathbf{A}^T| = 0$, by the above and rule v (or by question **1**). ■

b. $\mathbf{A}$ has two equal columns or two equal rows. *[1]*

Solution. Suppose $\mathbf{A}$ is a matrix whose ith and jth rows are the same (with $i \neq j$, of course). Then $\mathbf{A} \underset{R_i \leftrightarrow R_i}{\Longrightarrow} \mathbf{A}$, so, by rule *ii*, $|\mathbf{A}| = -|\mathbf{A}|$. The only number which is equal to its own negative is 0, so it must be the case that $|\mathbf{A}| = 0$.

By problem **1**, it also follows that $|\mathbf{A}| = 0$ if $\mathbf{A}$ has two equal columns. ■

c. $\mathbf{A}$ has rank less than n. *[1]*

Solution. One consequence of $\mathbf{A}$ having rank less than n is that the rows of $\mathbf{A}$ are dependent, so one row, say $\mathbf{r}_i$, is a linear combination of the other rows, say $\mathbf{r}_i = c_1\mathbf{r}_1 +$

[1] ... when rows are not in season!

$\cdots + c_{i-1}\mathbf{r}_{i-1} + c_{i+1}\mathbf{r}_{i+1} + \cdots + c_n\mathbf{r}_n$. If you proceed to modify $\mathbf{A}$ by successively subtracting c_j times row j from row i for $j = 1, \ldots, i-1, i+1, \ldots n$, you will get aall 0s in row i. Rule *iv* guarantees that the determinant will not change and, by the solution to **2a**, the determinant of the resulting matrix is 0. Hence $|\mathbf{A}| = 0$. ■

4 **3.** Use the Gauss-Jordan method to put each of the matrices below in reduced row-echelon form and then apply what you have learned above to use this computation to find their determinants.

a. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ *[1]* **b.** $\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 3 & 3 & 15 \end{bmatrix}$ *[3]*

Solution to **a**. Gauss-Jordan it is!

$$\begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \underset{R_2 - 3R_1}{\Longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \underset{-\frac{1}{2}R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \overset{R_1 - 2R_2}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we track how the determinant changed in the course of these operations until we reached the identity matrix:

$$|\mathbf{A}| \underset{(ii)}{\overset{\frac{1}{3}R_1}{\Longrightarrow}} -|\mathbf{A}| \underset{(iv)}{\overset{R_2-3R_1}{\Longrightarrow}} -|\mathbf{A}| \underset{(iii)}{\overset{-\frac{1}{2}R_2}{\Longrightarrow}} -\frac{1}{2}\left(-|\mathbf{A}|\right) = \frac{1}{2}|\mathbf{A}| \underset{(iv)}{\overset{R_1-2R_2}{\Longrightarrow}} \frac{1}{2}|\mathbf{A}| = |\mathbf{I}_2| \underset{(i)}{=} 1$$

It follows that $|\mathbf{A}| = 1 \div \frac{1}{2} = 2$. ■

Solution to **b**. First, we use the Gauss-Jordan method to put the matrix in reduced echelon form:

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 3 & 3 & 15 \end{bmatrix} \overset{R_1 \leftrightarrow R_2}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 2 & 3 & 4 & 1 \\ 3 & 3 & 3 & 15 \end{bmatrix} \underset{\substack{R_3 - 2R_1 \\ R_4 - 3R_1}}{\Longrightarrow} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -2 & -7 \\ 0 & -3 & -6 & 3 \end{bmatrix}$$

$$\underset{\substack{R_3 + R_2 \\ R_4 + 3R_2}}{\overset{R_1 - 2R_2}{\Longrightarrow}} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -7 \\ 0 & 0 & 3 & 3 \end{bmatrix} \underset{(-1)R_3}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 3 & 3 \end{bmatrix} \underset{R_4 - 3R_3}{\overset{\substack{R_1 - R_3 \\ R_2 - R_3}}{\Longrightarrow}} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & -18 \end{bmatrix}$$

$$\underset{-\frac{1}{18}R_4}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \overset{\substack{R_1 + 3R_4 \\ R_2 + 7R_4 \\ R_3 - 7R_4}}{\Longrightarrow} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Second, we track how the determinant changed in the course of these operations until we reached the identity matrix. For brevity, we'll omit the steps at which multiples of one row were added or subtracted from other rows, since by rule *iv*, these will not change the determinant.

$$|\mathbf{A}| \underset{(ii)}{\overset{R_1 \leftrightarrow R_2}{\Longrightarrow}} -|\mathbf{A}| \underset{(iii)}{\overset{(-1)R_3}{\Longrightarrow}} (-1)\left(-|\mathbf{A}|\right) = |\mathbf{A}| \underset{(iii)}{\overset{-\frac{1}{18}R_3}{\Longrightarrow}} \left(-\frac{1}{18}\right)|\mathbf{A}| = |\mathbf{I}_3| \underset{(i)}{=} 1$$

Solving for $|\mathbf{A}|$ at the very end gives $|\mathbf{A}| = 1 \div \left(-\frac{1}{18}\right) = -18$. ■