Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2009

Solutions to Assignment #3

Circular reasoning?

1. In two dimensions, the equation of a circle of radius r and with centre at (a, b) is $(x-a)^2 + (y-b)^2 = r^2$. Using algebra — including linear algebra! — find the radius and centre of the circle passing through the points $(1, 0)$, $(4, 3)$, and $(5, 2)$. *[4]*

SOLUTION. We plug in the three given points for (x, y) in the generic equation for a circle and rearrange the resulting equations a little:

$$\begin{cases} (1-a)^2 + (0-b)^2 = r^2 \\ (4-a)^2 + (3-b)^2 = r^2 \\ (5-a)^2 + (2-b)^2 = r^2 \end{cases}$$
$$\implies \begin{cases} 1 - 2a + a^2 + b^2 = r^2 \\ 16 - 8a + a^2 + 9 - 6b + b^2 = r^2 \\ 25 - 10a + a^2 + 4 - 4b + b^2 = r^2 \end{cases}$$
$$\implies \begin{cases} 2a - 1 = a^2 + b^2 - r^2 \\ 8a + 6b - 25 = a^2 + b^2 - r^2 \\ 10a + 4b - 29 = a^2 + b^2 - r^2 \end{cases}$$

Since the three right-hand sides of the rearranged equations are the same, the left-hand sides must be equal to one another. Setting each of the last two left-hand sides equal to the first gives us two linear equations in two unknowns each, namely a and b:

$$\begin{cases} 8a + 6b - 25 = 2a - 1 \\ 10a + 4b - 29 = 2a - 1 \end{cases} \implies \begin{cases} 6a + 6b = 24 \\ 8a + 4b = 28 \end{cases}$$

We solve this system for a and b using Gauss-Jordan reduction:

$$\left[\begin{array}{cc|c} 6 & 6 & 24 \\ 8 & 4 & 28 \end{array}\right] \overset{\frac{1}{6}R_1}{\implies} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 8 & 4 & 28 \end{array}\right] \underset{R_2 - 8R_1}{\implies} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & -4 & -4 \end{array}\right]$$
$$\underset{-\frac{1}{4}R_2}{\implies} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 0 & 1 & 1 \end{array}\right] \overset{R_1 - R_2}{\implies} \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

Thus $a = 3$ and $b = 1$, *i.e.* the centre of the (unique!) circle passing through the given points is $(3, 1)$.

We find r^2 by plugging in the coordinates of our centre, $a = 3$ and $b = 1$, and one of the given points, say $(1, 0)$, into the the generic equation for a circle, $(x-a)^2+(y-b)^2 = r^2$, and then solving for r^2:

$$(1-3)^2 + (1-0)^2 = r^2 \implies 2^2 + 1^2 = r^2 \implies r^2 = 5$$

Thus the unique circle passing through the three given points consists of all points (x, y) satisfying the equation $(x-3)^2 + (y-1)^2 = 5$. It follows that the radius of the circle is $r = \sqrt{5}$. ■

2. In general, do three points in two-dimensional space specify a circle passing through all three? If so, why? If not, why not? (You do not need to give a formal proof here, just a convincing argument.) *[4]*

SOLUTION. Given any three different points in two-dimensional space, one can try to execute the procedure used in the solution to problem **1**. It ought to work (and does!) so long as the points aren't positioned in a way that makes it impossible for them all to be on a circle; namely, so long as they are not all on the same line.

Since circles (of finite radius!) are not straight, it obviously impossible for three different points on a straight line to be on the same circle.

On the other hand, suppose that we have three points, say (p, q), (s, t), and (u, v), which are *not* on the same line. If we follow the same procedure used in the solution to problem **1** above, we get down to a system of equations equivalent to:

$$\begin{aligned} 2(p-s)a + 2(q-t)b &= (p^2+q^2) - (s^2+t^2) \\ 2(s-u)a + 2(t-v)b &= (s^2+t^2) - (u^2+v^2) \\ 2(u-p)a + 2(v-q)b &= (u^2+v^2) - (p^2+q^2) \end{aligned}$$

It's pretty easy to see that any two of these equations give you the third (just add the two you picked and multiply by -1), so we only need the first two. The matrix representation of the system given by the first two equations is:

$$\begin{bmatrix} 2(p-s) & 2(q-t) \\ 2(s-u) & 2(t-v) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} (p^2+q^2) - (s^2+t^2) \\ (s^2+t^2) - (u^2+v^2) \end{bmatrix}$$

This system has an unique solution if the matrix $\begin{bmatrix} 2(p-s) & 2(q-t) \\ 2(s-u) & 2(t-v) \end{bmatrix}$ is invertible, which will be the case if the column vectors $\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix}$ and $\begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix}$ are linearly independent. This boils down to checking whether one of the two is a multiple of the other.

Suppose, for the sake of argument, that there was a number m such that

$$\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix} = m \begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix}.$$

Writing this out coordinate by coordinate and cancelling those pesky 2s gives us:

$$p - s = m(q-t) \quad \text{and} \quad s - u = m(t-v)$$

Geometrically, this means that the line joining (p, q) and (s, t) has the same slope as the line joining (s, t) and (u, v); since these lines have the same slope and both pass through

(s,t), they are the same line and all three points are on it. *Oops!* That cannot be: we are considering the case where the three points are *not* on the same line. So $\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix}$ and $\begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix}$ can't be multiples of one another, which means they must be linearly independent.

Since $\begin{bmatrix} 2(p-s) \\ 2(s-u) \end{bmatrix}$ and $\begin{bmatrix} 2(q-t) \\ 2(t-v) \end{bmatrix}$ are linearly independent, $\begin{bmatrix} 2(p-s) & 2(q-t) \\ 2(s-u) & 2(t-v) \end{bmatrix}$ is invertible, which means that the associated system of linear equations has an unique solution for a and b. This gives us the centre of the circle, and we can find r as in the solution to problem **1**.

You can try to figure out what happens if the three points aren't all different, since this solution is already serious overkill ... (A *lot* less than the argument given above would have received full credit.) ■

3. How many points does one need in three-dimensional space to specify a sphere? What restriction(s) must there be on these points? *[2]*

Solution. In three-dimensional space, one can also use the same kind of procedure used in the solution to problem **1**. Three different points are not enough, though: think about resting balls of different sizes on the tops of three pillars spaced in an equilateral triangle.

Four points will do, though, so long as the points aren't positioned in a way that makes it impossible for them all to be on a unique sphere; namely, so long as they are not all on the same plane. Note that some configurations of four points in a plane, such as a square, are on many spheres, while other configurations of four points in a plane, such as a triangle with a point in the middle, are on no sphere's at all. ■