Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2009

Solutions to Assignment #3
Linear optimization

In this assignment we will deal with the solid whose faces are (parts of) the planes given by the equations $x = 0$, $y = 0$, $z = 0$, $-2x+y+z = 3$, $x-2y+z = 3$, $x+y-2z = 3$, $-3x + 2y + 2z = 7$, $2x - 3y + 2z = 7$, and $2x + 2y - 3z = 7$. Another way to look at this solid is as the set of points with coordinates (x, y, z) which satisfy *all* of the following nine inequalities: $x \geq 0$, $y \geq 0$, $z \geq 0$, $-2x + y + z \leq 3$, $x - 2y + z \leq 3$, $x + y - 2z \leq 3$, $-3x + 2y + 2z \leq 7$, $2x - 3y + 2z \leq 7$, and $2x + 2y - 3z \leq 7$.

1. Find the coordinates of all of the vertices of this solid and make as accurate a sketch as you can of it. *[6]*

SOLUTION. The vertices of the solid will be the points where three or more of the given planes intersect and which satisfy all of the given inequalities. Unfortunately, with nine planes, we have a *lot* of potential points of intersection to check for. (With nine planes, there are $\binom{9}{3} = 84$ possible ways to choose a subset of three ...) There are three natural groups of three planes each; the equations of the planes in each group use the same two coefficients and the same right-hand side, and differ only in cycling one of the two coefficients through the three different variables:

$$\begin{array}{ccc} x + 0y + 0z = 0 & -2x + y + z = 3 & -3x + 2y + 2z = 7 \\ 0x + y + 0z = 0 & x - 2y + z = 3 & 2x - 3y + 2z = 7 \\ 0x + 0y + z = 0 & x + y - 2z = 3 & 2x + 2y - 3z = 7 \end{array}$$

This symmetry will allow us to get some calculations for free once others have been done.

We will consider each group of three planes by itself first, and consider various combinations of planes from different groups later.

i. The three coordinate planes $x = 0$, $y = 0$, and $z = 0$. These obviously intersect in the one and only point with coordinates all 0s: $(0, 0, 0)$. This point survives all nine inequalities: $0 \geq 0$, so $x \geq 0$, $y \geq 0$, and $z \geq 0$ are satisfied. $-2 \cdot 0 + 0 + 0 = 0 \leq 3$, taking care of $-2x + y + z = 3$; $x - 2y + z \leq 3$ and $x + y - 2z \leq 3$ are satisfied in the same way. Finally, $-3 \cdot 0 + 2 \cdot 0 + 2 \cdot 0 = 0 \leq 7$, taking care of $-3x + 2y + 2z \leq 7$; $2x - 3y + 2z \leq 7$ and $2x + 2y - 3z \leq 7$ are satisfied in the same way. Hence $(0, 0, 0)$ is a vertex of the solid.

ii. The three planes $-2x + y + z = 3$, $x - 2y + z = 3$, and $x + y - 2z = 3$. We set up and solve the corresponding system of equations in augmented matrix form:

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 3 \\ 1 & -2 & 1 & 3 \\ 1 & 1 & -2 & 3 \end{array}\right] \begin{array}{c} R_1 \leftrightarrow R_2 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ -2 & 1 & 1 & 3 \\ 1 & 1 & -2 & 3 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 + 2R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & -3 & 3 & 9 \\ 0 & 3 & -3 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{3}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 3 & -3 & 0 \end{array}\right] \begin{array}{c} R_1 + 2R_2 \\ \Longrightarrow \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 9 \end{array}\right]$$

Since, in the final matrix, row three is all zeroes in the coefficient columns and non-zero in the right-hand column, *i.e.* it represents $0x+0y+0z=9$, there is no solution, so the three planes do not have any common point of intersection.

iii. The three planes $-3x+2y+2z=7$, $2x-3y+2z=7$, and $2x+2y-3z=7$. We set up and solve the corresponding system of equations in augmented matrix form:

$$\left[\begin{array}{ccc|c} -3 & 2 & 2 & 7 \\ 2 & -3 & 2 & 7 \\ 2 & 2 & -3 & 7 \end{array}\right] \begin{array}{c} -\frac{1}{3}R_1 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{7}{3} \\ 2 & -3 & 2 & 7 \\ 2 & 2 & -3 & 7 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2-2R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{7}{3} \\ 0 & -\frac{5}{3} & \frac{10}{3} & \frac{35}{3} \\ 0 & \frac{10}{3} & -\frac{5}{3} & \frac{35}{3} \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{3}{5}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{2}{3} & -\frac{7}{3} \\ 0 & 1 & -2 & -7 \\ 0 & \frac{10}{3} & -\frac{5}{3} & \frac{35}{3} \end{array}\right] \begin{array}{c} R_1+\frac{2}{3}R_2 \\ \Longrightarrow \\ R_3-\frac{10}{3}R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 5 & 35 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ \frac{1}{5}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -7 \\ 0 & 1 & -2 & -7 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

$$\begin{array}{c} R_1+2R_3 \\ R_2+2R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 7 \end{array}\right]$$

Thus the three planes intersect in the unique point $(7,7,7)$. This point survives all nine inequalities: $7 \geq 0$, so $x \geq 0$, $y \geq 0$, and $z \geq 0$ are satisfied. $-2\cdot 7+7+7=0 \leq 3$, taking care of $-2x+y+z=3$; $x-2y+z \leq 3$ and $x+y-2z \leq 3$ are satisfied in the same way. Finally, $-3\cdot 7+2\cdot 7+2\cdot 7=7 \leq 7$, taking care of $-3x+2y+2z \leq 7$; $2x-3y+2z \leq 7$ and $2x+2y-3z \leq 7$ are satisfied in the same way. Hence $(7,7,7)$ is a vertex of the solid.

3 combinations of three planes down, 81 to go! We now begin considering combinations of planes from different groups. We'll save a bit more effort by considering some combinations of four, where those four intersect in the same point. (Hindsight is sharper than foresight!)

iv. The four planes $x=0$, $y=0$, $-2x+y+z=3$, and $x-2y+z=3$. If we're in a bit of a rut, we set up the augmented matrix and go:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 1 & 1 & 3 \\ 1 & -2 & 1 & 3 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ R_3+2R_1 \\ R_4-R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 \\ 0 & -2 & 1 & 3 \end{array}\right] \begin{array}{c} \\ \Longrightarrow \\ R_3-R_2 \\ R_4+2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ \\ R_4-R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Hence the four planes intersect in one common point, namely $(0,0,3)$. If we're a bit smarter, we simply notice that if we know that $x=0$ and $y=0$, we can simply plug this in and solve for z in both $-2x+y+z=3$ and $x-2y+z=3$, getting $z=3$ in each case.

Either way, this point survives all nine inequalities: $0 \geq 0$, so $x \geq 0$ and $y \geq 0$ are satisfied; and $z \geq 0$ is satisfied because $3 \geq 0$. $-2\cdot 0+0+3=0+-2\cdot 0+3=3 \leq 3$, taking care of $-2x+y+z \leq 3$ and $x-2y+z \leq 3$, and $x-2y+z \leq 3$ is satisfied because $0+0-2\cdot 3=-6\leq 3$. Finally, $-3\cdot 0+2\cdot 0+2\cdot 3=2\cdot 0+-3\cdot 0+2\cdot 3=6\leq 7$, taking care of $-3x+2y+2z\leq 7$ and $2x-3y+2z\leq 7$, while $2x+2y-3z\leq 7$ is satisfied because $2\cdot 0+2\cdot 0-3\cdot 3=-9\leq 7$. Hence $(0,0,3)$ is a vertex of the solid.

Observe that if we interchange the variables x and z in the equations of the planes we dealt with above, we get the equations $z=0$, $y=0$, $-2z+y+x=3$, and $z-2y+x=3$ (*i.e.* $y=0$, $z=0$, $x+y-2z=3$, and $x-2y+z=3$ if we write the variables in their usual order). These have exactly the same form as the equations in *iv* and will give rise to the same calculation, with the same result, as in *iv*, with the roles of x and z interchanged. Thus the planes given by the equations $y=0$, $z=0$, $x+y-2z=3$, and $x-2y+z=3$ intersect in the single point $(3,0,0)$ (*i.e.* $(0,0,3)$ with the x- and z-coordinates interchanged). Since swapping the roles of x and z in the nine inequalities gives you back the same nine inequalities, $(3,0,0)$ will satisfy all of them because $(0,0,3)$ does. Hence $(3,0,0)$ is also a vertex of the solid.

In the same way, if we interchange the roles of y and z equations of the planes we dealt with above, we get the equations $x=0$, $z=0$, $-2x+y+x=3$, and $x+y-2z=3$, and the planes given by these equations intersect in the point $(0,3,0)$, which also turns out to be a vertex of the solid.

Note that interchanging the roles of x and y in the equations of the planes just gives us the same four equations back, contributing nothing new. Nevertheless, we have taken care of $3\binom{4}{3}=3\cdot 4=12$ combinations of three of the nine planes and found three vertices of the solid with just one calculation and check of its result against the nine inequalities.

$3+12=15$ combinations of three planes down, 69 to go!

v. The three planes $x=0$, $y=0$, and $x+y-2z=3$. Plugging $x=y=0$ into the last equation and solving for z gives us $z=-\frac{3}{2}$, *i.e.* these planes intersect in a single point, namely $\left(0,0,-\frac{3}{2}\right)$. This point is not in the solid: since $-\frac{3}{2}<0$, it fails the inequality $z\geq 0$.

Interchanging the roles of x and z now gives us the point $\left(-\frac{3}{2},0,0\right)$ as the single point of intersection of $y=0$, $z=0$, and $-2x+y+z=3$; which point is also not in the solid, failing $x\geq 0$. Similarly, interchanging the roles of y and z instead gives us the point $\left(0,-\frac{3}{2},0\right)$ as the single point of intersection of $x=0$, $z=0$, and $x-2y+z=3$; which point is also not in the solid, failing $y\geq 0$.

$15+3=18$ combinations of three planes down, 66 to go!

vi. The four planes $x=0$, $y=0$, $-3x+2y+2z=7$, and $2x-3y+2z=7$. If we know that $x=0$ and $y=0$, we can simply plug this in and solve for z in both $-3x+2y+2z=7$ and $2x-3y+2z=7$, getting $z=\frac{7}{2}$ in each case. Thus these four planes intersect in the single point $\left(0,0,\frac{7}{2}\right)$. This point is not a vertex of the solid because $-2\cdot 0+\cdot 0+\frac{7}{2}=\frac{7}{2}>3$, thus failing the inequality $-2x+y+z\leq 3$.

By interchanging the roles of x and y with z in the above, we also get that $y=0$, $z=0$, $2x+2y-3z=7$, and $2x-3y+2z=7$ all intersect in $\left(\frac{7}{2},0,0\right)$, and that $x=0$, $z=0$, $-3x+2y+2z=7$, and $2x+2y-3z=7$ all intersect in $\left(0,\frac{7}{2},0\right)$. Neither of these points is a vertex either, failing the inequalities $x+y-2z\leq 3$ and $-2x+y+z\leq 3$, respectively.

All this takes care of twelve more possible combinations of three of the nine planes. $18 + 12 = 30$ combinations of three planes down, 54 to go!

vii. The three planes $x = 0$, $y = 0$, and $2x + 2y - 3z = 7$. Plugging $x = y = 0$ into the last equation and solving for z gives us $z = -\frac{7}{3}$, *i.e.* these planes intersect in a single point, namely $\left(0, 0, -\frac{7}{3}\right)$. This point is not in the solid: since $-\frac{7}{3} < 0$, it fails the inequality $z \geq 0$.

Interchanging the roles of x and z now gives us the point $\left(-\frac{7}{3}, 0, 0\right)$ as the single point of intersection of $y = 0$, $z = 0$, and $-3x + 2y + 2z = 7$; which point is also not in the solid, failing $x \geq 0$. Similarly, interchanging the roles of y and z instead gives us the point $\left(0, -\frac{7}{3}, 0\right)$ as the single point of intersection of $x = 0$, $z = 0$, and $2x - 3y + 2z = 7$; which point is also not in the solid, failing $y \geq 0$.

$30 + 3 = 33$ combinations of three planes down, 51 to go!

viii. The three planes $x = 0$, $x + y - 2z = 3$, and $2x + 2y - 3z = 7$. Plugging in $x = 0$ into the other two equations leaves us with two equations in two unknowns, $y - 2z = 3$ and $2y - 3x = 7$. We solve these by applying the Gauss-Jordan algorithm to the corresponding augmented matrix. (It's overkill, but what the heck ...)

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -3 & 7 \end{array}\right] \underset{R_2 - 2R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 1 \end{array}\right] \underset{\Longrightarrow}{R_1 + 2R_2} \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \end{array}\right]$$

Thus these planes intersect in a single point, namely $(0, 5, 1)$. This point is not in the solid: since $-3 \cdot 0 + 2 \cdot 5 + 2 \cdot 1 = 12 > 7$, it fails the inequality $-3x + 2y + 2z \leq 7$.

Interchanging the roles of x and y gives us $(5, 0, 1)$ as the single point of intersection of $y = 0$, $x + y - 2z = 3$, and $2x + 2y - 3z = 7$. This point is also not in the solid, failing the inequality $2x - 3y + 2z \leq 7$. Similarly, interchanging the roles of x and z gives $(1, 5, 0)$ as the single point of intersection of $z = 0$, $-2x + y + z = 3$, and $-3x + 2y = 2z = 7$, and this point is also not in the solid, failing the inequality $2x + 2y - 3z \leq 7$. Also, interchanging the roles of y and z gives us $(0, 1, 5)$ as the single point of intersection of $x = 0$, $x - 2y + z = 3$, and $2x - 3y + 2z = 7$, and this point is also not in the solid, failing the inequality $-3x + 2y + 2z \leq 7$.

One can go further here by swapping all three of the variables around instead of just two at a time. Having y take on the role of x, z take on the role of y, and x take on the role of z gives us $(1, 0, 5)$ as the single point of intersection of $y = 0$, $-2x + y + z$, and $-3x + 2y + 2z = 0$. This point is also not in the solid, failing the inequality $2x - 3y + 2z \leq 7$. Similarly, having z take on the role of x, y take on the role of z, and x take on the role of y gives us $(5, 1, 0)$ as the single point of intersection of $z = 0$, $x - 2y + z = 3$, and $2x - 3y + 2z = 7$, and this point is also not in the solid, failing the inequality $2x + 2y - 3z \leq 7$.

$33 + 6 = 39$ combinations of three planes down, 45 to go!

ix. The three planes $x = 0$, $-2x + y + z = 3$, and $-3x + 2y + 2z = 7$. Plugging in $x = 0$ into the other two equations leaves us with two equations in two unknowns, $y + z = 3$ and $2y + 2z = 7$. Since the first of these implies that $2y + 2z = 2(y + z) = 2 \cdot 3 = 6 \neq 7$, these equations are mutually inconsistent and have no solution. Hence the three planes do not meet in any common point.

Interchanging the roles of x and y tells us that $y = 0$, $x - 2y + z = 3$, and $2x - 3y + 2z = 7$ also do not have a common point. Similarly, interchanging the roles of x and z tells us that $z = 0$, $x + y - 2z = 0$, and $2x + 2y - 3z = 7$ do not have a common point.

$39 + 3 = 42$ combinations of three planes down, 42 to go! Halfway there!

x. The four planes $x-2y+z = 3$, $-2x+y+z = 3$, $2x-3y+2z = 7$, and $-3x+2y+2z = 7$. We set up the augmented matrix and go:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ -2 & 1 & 1 & 3 \\ 2 & -3 & 2 & 7 \\ -3 & 2 & 2 & 7 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 + 2R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & -3 & 3 & 9 \\ 0 & 1 & 0 & 1 \\ 0 & -4 & 5 & 16 \end{array}\right]$$

$$\begin{array}{c} R_2 \leftrightarrow R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & -3 & 3 & 9 \\ 0 & -4 & 5 & 16 \end{array}\right] \begin{array}{c} R_1 + 2R_2 \\ \Longrightarrow \\ R_3 + 3R_2 \\ R_4 + 4R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 5 & 20 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ \frac{1}{3}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 5 & 20 \end{array}\right] \begin{array}{c} R_1 - R_3 \\ \\ \Longrightarrow \\ R_4 - 5R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This tells us that the four planes $x - 2y + z = 3$, $-2x + y + z = 3$, $2x - 3y + 2z = 7$, and $-3x + 2y + 2z = 7$ intersect in a single common point, namely $(1, 1, 4)$.

This point satisfies all nine inequalities: $1 \geq 0$, so $x \geq 0$ and $y \geq 0$ are satisfied, and $4 \geq 0$, so $z \geq 0$ is also satisfied. $1 - 2 \cdot 1 + 4 = -2 \cdot 1 + 1 + 4 = 3 \leq 3$, so $x - 2y + z = 3$ and $-2x + y + z = 3$ are satisfied, and $1 + 1 - 2 \cdot 4 = -6 \leq 3$, so $x + y - 2z = 3$ is also satisfied. Finally, $2 \cdot 1 - 3 \cdot 1 + 2 \cdot 4 = -3 \cdot 1 + 2 \cdot 1 + 2 \cdot 4 = 7 \leq 7$, so $2x - 3y + 2z = 7$ and $-3x + 2y + 2z = 7$ are satisfied, and $2 \cdot 1 + 2 \cdot 1 - 3 \cdot 4 = -8 \leq 7$, so $2x + 2y - 3z = 7$ is also satisfied. It follows that $(1, 1, 4)$ is a vertex of the solid.

Interchanging the roles of x and z gives us $(4, 1, 1)$ as the single common point of intersection of the four planes $x - 2y + z = 3$, $x + y - 2z = 3$, $2x - 3y + 2z = 7$, and $2x + 2y - 3z = 7$. For the reasons given in *iv*, this point also satisfies all nine inequalities because $(1, 1, 4)$ does, and so is also a vertex of the solid.

Similarly, interchanging the roles of y and z gives us $(1, 4, 1)$ as the single common point of intersection of the four planes $x+y-2z = 3$, $-2x+y+z = 3$, $2x+2y-3z = 7$, and $-3x + 2y + 2z = 7$. For the reasons given in *iv*, this point also satisfies all nine inequalities because $(1, 1, 4)$ does, and so is also a vertex of the solid.

Note that interchanging the roles of x and y in the equations of the planes $x - 2y + z = 3$, $-2x + y + z = 3$, $2x - 3y + 2z = 7$, and $-3x + 2y + 2z = 7$ just gives us the same four equations back, contributing nothing new. Still, we did take care of $3\binom{4}{3} = 3 \cdot 4 = 12$ combinations of three of the nine planes and found three more vertices of the solid.

$42 + 12 = 54$ combinations of three planes down, 30 to go!

xi. The three planes $x-2y+z=3$, $-2x+y+z=3$, and $2x+2y-3z=7$. We set up the augmented matrix and go:

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ -2 & 1 & 1 & 3 \\ 2 & 2 & -3 & 7 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2+2R_1 \\ R_3-2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & -3 & 3 & 9 \\ 0 & 6 & -5 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{3}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 0 & 6 & -5 & 1 \end{array}\right] \begin{array}{c} R_1+2R_2 \\ \Longrightarrow \\ R_3-6R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 1 & 19 \end{array}\right]$$

$$\begin{array}{c} R_1+R_3 \\ R_2+R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 16 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 19 \end{array}\right]$$

It follows that the three planes $x-2y+z=3$, $-2x+y+z=3$, and $2x+2y-3z=7$ meet in just one point, namely $(16,16,19)$. This point is not in the solid because it fails the inequality $-3x+2y+2z \leq 7$, since $-3\cdot 16+2\cdot 16+2\cdot 19=35>7$.

Interchanging the roles of x and z gives us the point $(19,16,16)$ as the sole point of intersection of the three planes $x-2y+z=3$, $x+y-2z=3$, and $-3x+2y+2z=7$, and this point is also not in the solid because it fails the inequality $23x+2y-3z \leq 7$. Similalrly, interchanging the roles of y and x gives us the point $(16,19,16)$ as the sole point of intersection of the three planes $x+y-2z=3$, $-2x+y+z=3$, and $2x-3y+2z=7$, and this point is also not in the solid because it fails the inequality $-3x+2y+2z \leq 7$.

$54+3=57$ combinations of three planes down, 27 to go!

xii. The three planes $x+y-2z=3$, $2x-3y+2z=7$, and $-3x+2y+2z=7$. We set up the augmented matrix and go:

$$\left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 2 & -3 & 2 & 7 \\ -3 & 2 & 2 & 7 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2-2R_1 \\ R_3+3R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & -5 & 6 & 1 \\ 0 & 5 & -4 & 16 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{5}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & -2 & 3 \\ 0 & 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 5 & -4 & 16 \end{array}\right] \begin{array}{c} R_1-R_2 \\ \Longrightarrow \\ R_3-5R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{5} & \frac{16}{5} \\ 0 & 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 0 & 2 & 17 \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ \frac{1}{2}R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & -\frac{4}{5} & \frac{16}{5} \\ 0 & 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 0 & 1 & \frac{17}{2} \end{array}\right] \begin{array}{c} R_1+\frac{4}{5}R_3 \\ R_2+\frac{6}{5}R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & \frac{17}{2} \end{array}\right]$$

It follows that the three planes $x+y-2z=3$, $2x-3y+2z=7$, and $-3x+2y+2z=7$ intersect in the single point $\left(10,10,\frac{17}{2}\right)$. This point is not in the solid because it fails the inequality $2x+2y-3z \leq 7$, since $2\cdot 10+2\cdot 10-3\cdot\frac{17}{2}=\frac{29}{2}>7$.

Interchanging the roles of x and z gives us $\left(\frac{17}{2},10,10\right)$ as the sole point of intersection of the planes $-2x+y+z=3$, $2x-3y+2z=7$, and $2x+2y-3z=7$, and this point is also not in the solid because it fails the inequality $-3x+2y+2z \leq 7$.

Similarly, interchanging the roles of y and z gives us $\left(10, \frac{17}{2}, 10\right)$ as the sole point of intersection of the planes $x - 2y + z = 3$, $2x + 2y - 3z = 7$, and $-3x + 2y + 2z = 7$, and this point is also not in the solid because it fails the inequality $2x - 3y + 2z \leq 7$.

$57 + 3 = 60$ combinations of three planes down, 24 to go!

xiii. The three planes $x = 0$, $2x - 3y + 2z = 7$, and $2x + 2y - 3z = 7$. Plugging in $x = 0$ into the other two equations boils this down to $-3y + 2z = 7$ and $2y - 3z = 7$. We set up the augmented matrix for this smaller system and go:

$$\left[\begin{array}{cc|c} -3 & 2 & 7 \\ 2 & -3 & 7 \end{array}\right] \overset{R_1 + 2R_2}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -4 & 21 \\ 2 & -3 & 7 \end{array}\right] \underset{R_2 - 2R_1}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -4 & 21 \\ 0 & 5 & -35 \end{array}\right]$$

$$\underset{\frac{1}{5}R_2}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & -4 & 21 \\ 0 & 1 & -7 \end{array}\right] \overset{R_1 + 4R_2}{\Longrightarrow} \left[\begin{array}{cc|c} 1 & 0 & -7 \\ 0 & 1 & -7 \end{array}\right]$$

It follows that the three planes $x = 0$, $2x - 3y + 2z = 7$, and $2x + 2y - 3z = 7$ intersect in just one point, namely $(0, -7, -7)$. This point is not in the solid because it fails the inequality $z \geq 0$, since $-7 < 0$.

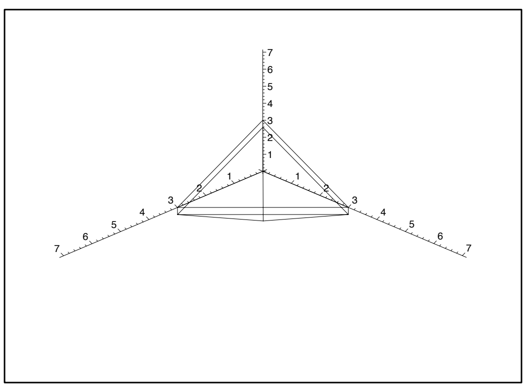
Interchanging the roles of x and z gives us $(-7, -7, 0)$ as the sole point of intersection of the planes $z = 0$, $2x - 3y + 2z = 7$, and $-3x + 2y + 2z = 7$, and this point is also not in the solid because it fails the inequality $x \geq 0$. Similarly, interchanging the roles of x and y gives us $(-7, 0, -7)$ as the sole point of intersection of the planes $y = 0$, $-3x + 2y + 2z = 7$, and $2x + 2y - 3z = 7$, and this point is also not in the solid because it fails the inequality $z \geq 0$.

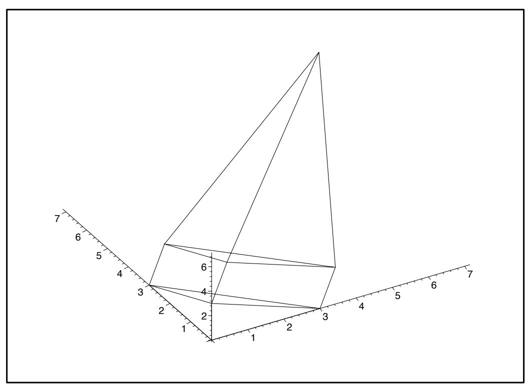
$60 + 3 = 63$ combinations of three planes down, 21 to go!

xiv. —-

Ah, nevermind! We leave the rest to the enterprising reader . . . :-)

The solid in question turns out to have eight vertices: $(0, 0, 0)$, $(3, 0, 0)$, $(0, 3, 0)$, $(0, 0, 3)$, $(4, 1, 1)$, $(1, 4, 1)$, $(1, 1, 4)$, and $(7, 7, 7)$. We can sketch the solid by plotting these points and connecting up adjacent ones. Here are two views of it, generated by `MAPLE`:





Here endeth the question! ■

2. Find the maximum value of the function $f(x, y, z) = 2x + 10y - 9z$ on this solid and determine at which point(s) of the solid this maximum occurs. *[4]*

SOLUTION. To find the maximum value of $f(x, y, z) = 2x + 10y - 9z$ on the solid, we check the value of $f(x, y, z)$ at each of the vertices of the solid:

$$\begin{aligned} f(0,0,0) &= 0 \\ f(3,0,0) &= 6 \\ f(0,3,0) &= 30 \\ f(0,0,3) &= -27 \\ f(4,1,1) &= 9 \\ f(1,4,1) &= 33 \\ f(1,1,4) &= -24 \\ f(7,7,7) &= 21 \end{aligned}$$

The largest of these, $f(1, 4, 1) = 33$, is the maximum value of $f(x, y, z) = 2x + 10y - 9z$ on the solid. To verify this, observe that since $x+y-2z \leq 3$, $-2x+y+z \leq 3$, $-3x+2y+2z \leq 7$, and $2x + 2y - 3z \leq 7$, for all points (x, y, z) in the solid,

$$\begin{aligned} f(x, y, z) = 2x + 10y - 9z &= 3(x + y - 2z) + (-2x + y + z) \\ &\qquad + (-3x + 2y + 2z) + 2(2x + 2y - 3z) \\ &\leq 3 \cdot 3 + 3 + 7 + 2 \cdot 7 = 33 \end{aligned}$$

for all points in the solid. Hence 33 is the largest value that $f(x, y, z)$ can achieve on the solid. ■