Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2009

Solutions to the Quizzes

Quiz #1. Friday, 24 September, 2009 *(10 minutes)*

Consider the line in two dimensions given by the equation $y = \frac{1}{2}x - 1$.

1. Find the points at which this line crosses the axes and sketch this line. *[2]*
2. Find a parametric equation(s) for this line. *[3]*

SOLUTION TO 1. To find the y-intercept, we set $x = 0$ in the equation of the line and solve for y:

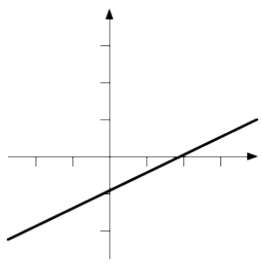
$$y = \frac{1}{2} \cdot 0 - 1 = 0 - 1 = 0$$

To find the x-intercept, we set $y = 0$ in the equation of the line and solve for x:

$$0 = \frac{1}{2}x - 1 \Longrightarrow \frac{1}{2}x = 1 \Longrightarrow x = 2 \cdot 1 = 2$$

Thus the line crosses the y-axis at the point $(0, -1)$ and the x-axis at the point $(2, 0)$.

Here's a sketch of the line:



SOLUTION TO 2. We need a base vector and a direction vector for the line. First, for the base vector we can take the vector that goes from the origin to a point on the line, say the x-intercept obtained above: $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$. Second, for the direction vector we can take the vector that takes us from one point on the line to another point on the line, say the one that takes us from the y-intercept to the x-intercept: $\begin{bmatrix} 2 - 0 \\ 0 - (-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

This means that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is a vector-parametric equation for the given line. One could also write this as a pair of parametric equations, one for each coordinate:

$$x = 2 + 2t$$
$$y = 0 + 1t = t \quad \blacksquare$$

Quiz #2. Friday, 2 October, 2009 *(5 minutes)*

Let $\mathbf{a} = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 5\sqrt{3} \end{bmatrix}$.

1. Find the lengths of **a** and **b**. *[2]*
2. Find the angle θ between **a** and **b**. *[3]*

Note: The quiz was presented in class with a typo. **b** was given as $\begin{bmatrix} 3 \\ 4 \\ \sqrt{3} \end{bmatrix}$ instead of $\begin{bmatrix} 3 \\ 4 \\ 5\sqrt{3} \end{bmatrix}$, which, sadly, makes a huge difference to how easily things work out. The solutions below are for what the question should have been ...

Solution to 1. We go to it:

$$\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{3^2+4^2+0^2} = \sqrt{9+16+0} = \sqrt{25} = 5$$

and

$$\|\mathbf{b}\| = \sqrt{\mathbf{b}\cdot\mathbf{b}} = \sqrt{3^2+4^2+\left(5\sqrt{3}\right)^2} = \sqrt{9+16+25\cdot 3} = \sqrt{100} = 10\,. \qquad \blacksquare$$

Solution to 2. Recall that $\mathbf{a}\cdot\mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos(\theta)$. Rearranging to solve for $\cos(\theta)$ and then computing away gives us:

$$\cos(\theta) = \frac{\mathbf{a}\cdot\mathbf{b}}{\|\mathbf{a}\|\|\mathbf{b}\|} = \frac{\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}\cdot\begin{bmatrix} 3 \\ 4 \\ 5\sqrt{3} \end{bmatrix}}{5\cdot 10} \qquad \text{(Since } \|\mathbf{a}\| = 5 \text{ and } \|\mathbf{b}\| = 10 \text{ from question 1.)}$$

$$= \frac{3\cdot 3+4\cdot 4+0\cdot 5\sqrt{3}}{50} = \frac{9+16+0}{50} = \frac{25}{50} = \frac{1}{2}$$

It follows that $\theta = 60^\circ = \frac{\pi}{3}$ radians. $\blacksquare$

Quiz #3. Friday, 9 October, 2009 *(10 minutes)*

Consider the plane given by the vector-parametric equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix},$$

where s and t are the parameters.

1. Find a normal vector for this plane. *[2]*
2. Find an equation of the form $ax + by + cz = d$ describing this plane. *[2]*

SOLUTION TO 1. A vector normal to this plane would be one that is orthogonal to both of the direction vectors used in the parametrization. The cross-product of these two vectors is such a vector, so we compute it:

$$\begin{aligned}
\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -1 & 0 \\ 0 & -1 & 3 \end{vmatrix} = \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} \mathbf{k} \\
&= ((-1) \cdot 3 - 0 \cdot (-1))\,\mathbf{i} - (2 \cdot 3 - 0 \cdot 0)\,\mathbf{j} + (2 \cdot (-1) - (-1) \cdot 0)\,\mathbf{k} \\
&= -3\mathbf{i} - 6\mathbf{j} - 2\mathbf{k} = \begin{bmatrix} -3 \\ -6 \\ -2 \end{bmatrix} \qquad \blacksquare
\end{aligned}$$

SOLUTION TO 2. We can use the coordinates of any vector normal to the plane as the coefficients a, b, and c. In this case, we can use the normal vector obtained in the solution to question 1, giving us $a = -3$, $b = -6$, and $c = -2$. To obtain d, we can plug the coordinates of any point on the plane into $-3x - 6y - 2z = d$ and solve for d. The base point of the given plane, $(0, 1, 0)$, is convenient in this case: $d = -3 \cdot 0 - 6 \cdot 1 - 2 \cdot 0 = -6$.

Thus an equation of the form $ax + by + cz = d$ describing the given plane is $-3x - 6y - 2z = -6$. Those who dislike negative numbers can multiply through by -1 to get the equation $3x + 6y + 2z = 6$, which also describes the given plane. $\blacksquare$

Quiz #4. Friday, 16 October, 2009 *(10 minutes)*

1. Find the point(s), if any, in which the planes given by the equations

$$\begin{array}{rcrcrcl} x & + & y & + & z & = & 1 \\ 3x & - & y & + & z & = & 1 \\ x & - & y & & & = & 0 \end{array}$$

intersect. *[5]*

SOLUTION. We set up the augmented matrix corresponding to the system of linear equations we get by considering the equations of the planes simultaneously, and apply the Gauss-Jordan algorithm.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & -1 & 1 & 1 \\ 1 & -1 & 0 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - 3R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & -2 & -2 \\ 0 & -2 & -1 & -1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{4}R_2 \\ \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -2 & -1 & -1 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 + 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We halt here since we've reached "reduced row echelon form." There are two surviving rows that aren't all 0s, corresponding to the linear equations:

$$\begin{array}{rcrcrcl} x & & & + & \frac{1}{2}z & = & \frac{1}{2} \\ & & y & + & \frac{1}{2}z & = & \frac{1}{2} \end{array}$$

This means we have (infinitely!) many solutions, since we can solve for x and y no matter what value we plug in for z. If we let $z = t$ for a parameter t, and solve for x and y in terms of t, we get the following parametric equations for the solutions:

$$\begin{aligned} x &= \frac{1}{2} - \frac{1}{2}t \\ y &= \frac{1}{2} - \frac{1}{2}t \\ z &= t \end{aligned}$$

It follows that the set of points in which all three of the given planes intersect consists of the line with base vector $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}$ and direction vector $\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$. ■

Quiz #5. Friday, 23 October, 2009 *(10 minutes)*

1. Determine whether $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is in $\text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}\right\}$ or not. *[5]*

SOLUTION. We need to discover whether or not there are scalars a, b, and c such that

$$a\begin{bmatrix}1\\1\\1\end{bmatrix} + b\begin{bmatrix}1\\2\\-1\end{bmatrix} + c\begin{bmatrix}3\\4\\1\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}.$$

Considering this vector equation coordinate by coordinate, this boils down to asking whether we can solve the following system of linear equations:

$$\begin{array}{ccccccc} a & + & b & + & 3c & = & 1 \\ a & + & 2b & + & 4c & = & 2 \\ a & - & b & + & c & = & 3 \end{array}$$

To do this, we set up the corresponding augmented matrix and use the Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 1 & 2 & 4 & 2 \\ 1 & -1 & 1 & 3 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -2 & -2 & 2 \end{array}\right] \begin{array}{c} R_1 - R_2 \\ \Longrightarrow \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

Since the third row now has 0s for the coefficients and a something other than zero in the last column, *i.e.* it corresponds to the unsolvable equation $0a + 0b + 0c = 4$, there are no solutions to the original system of linear equations. It follows that $\begin{bmatrix}1\\2\\3\end{bmatrix}$ is not in $\text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\-1\end{bmatrix}, \begin{bmatrix}3\\4\\1\end{bmatrix}\right\}$. ■

Quiz #6. Friday, 13 November, 2009 *(10 minutes)*

1. Why is there only one 2×2 matrix $\mathbf{A}$ such that $\mathbf{BA} = \mathbf{B}$ for every 2×2 matrix $\mathbf{B}$? *[5]*

SOLUTION 1. *(With a bit of abstraction ...)* Suppose $\mathbf{A}$ is a 2×2 matrix such that $\mathbf{BA} = \mathbf{B}$ for every 2×2 matrix $\mathbf{B}$. We will show that $\mathbf{A} = \mathbf{I}_2$. First, plugging in $\mathbf{I}_2$ for $boldB$, we get $\mathbf{I}_2\mathbf{A} = \mathbf{I}_2$. Second, $\mathbf{I_2 A} = \mathbf{A}$. It follows that $\mathbf{A} = \mathbf{I}_2$, so there is only one matrix, namely $\mathbf{I}_2$, satisfying the given condition. ■

SOLUTION 2. *(With a bit of brute farce – er, force ...)* Suppose $\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix such that $\mathbf{BA} = \mathbf{B}$ for every 2×2 matrix $\mathbf{B}$. Plugging in $\mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ for $\mathbf{B}$ gives us

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1a + 0b & 0a + 1b \\ 1c + 0d & 0c + 1d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

i.e. $a = 1$, $b = 0$, $c = 0$, and $d = 1$, which is the only possible solution. Thus

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{I}_2 ,$$

and so there is only one such $\mathbf{A}$. ■

Quiz #7. Friday, 20 November, 2009 *(10 minutes)*

1. Find the inverse of $\mathbf{A} = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}$ or show that it is not invertible. *[5]*

SOLUTION. We attempt to invert the matrix using the Gauss-Jordan algorithm on the "super-augmented" matrix.

$$\left[\begin{array}{ccc|ccc} 3 & 2 & 1 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 1 & 3 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 \leftrightarrow R_3 \\ \Longrightarrow \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 2 & 1 & 3 & 0 & 1 & 0 \\ 3 & 2 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & -5 & -1 & 0 & 1 & -2 \\ 0 & -7 & -5 & 1 & 0 & -3 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ -\frac{1}{5}R_2 \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 2 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{5} & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & -7 & -5 & 1 & 0 & -3 \end{array}\right]$$

$$\begin{array}{c} R_1 - 3R_2 \\ \Longrightarrow \\ R_3 + 7R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{5} & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{5} & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & -\frac{18}{5} & 1 & -\frac{7}{5} & -\frac{1}{5} \end{array}\right]$$

$$\begin{array}{c} \\ \Longrightarrow \\ -\frac{5}{18}R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{7}{5} & 0 & \frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & \frac{1}{5} & 0 & -\frac{1}{5} & \frac{2}{5} \\ 0 & 0 & 1 & -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{array}\right]$$

$$\begin{array}{c} R_1 - \frac{7}{5}R_3 \\ R_2 - \frac{1}{5}R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ 0 & 1 & 0 & \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ 0 & 0 & 1 & -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{array}\right]$$

It follows that $\mathbf{A}$ is invertible and that $\mathbf{A}^{-1} = \begin{bmatrix} \frac{7}{18} & \frac{1}{18} & -\frac{5}{18} \\ \frac{1}{18} & -\frac{5}{18} & \frac{7}{18} \\ -\frac{5}{18} & \frac{7}{18} & \frac{1}{18} \end{bmatrix}$. ■

Quiz #8. Friday, 27 November, 2009 *(10 minutes)*

$$\text{Let } \mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & -1 & -2 \end{bmatrix}.$$

1. Find a basis for the null space of **A**. *[4]*
2. Use your work for problem 1 to identify a basis of the column space of **A**. *[1]*

SOLUTION TO 1. The null space of **A** consists of all vectors **x** satisfying the equation $\mathbf{Ax} = \mathbf{0}$. We solve this by setting up the corresponding augmented matrix and reducing it using the Gauss-Jordan algorithm:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \underset{R_2+R_1}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -1 & -2 & 0 \end{array}\right] \underset{R_3+R_2}{\Longrightarrow} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This last corresponds to the linear system $x + 2z = 0$ and $y + 2z = 0$. It is easy to describe all the solutions to this system parametrically. If we set $z = t$, where t is the parameter, then $x = -2t$ and $y = -2t$. Thus the null space of **A** is

$$\operatorname{null}(\mathbf{A}) = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \text{ for } t \in \mathbb{R} \right\} = \operatorname{Span} \left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Thus $\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of **A**. ■

SOLUTION TO 2. Looking at the row-reduced matrix we obtained in our solution to question 1, we note that the leading 1s in the non-zero rows occur in columns 1 and 2. It follows that columns 1 and 2 of the original matrix **A** form a basis for its column space:

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}. \qquad ■$$

Quiz #9. Friday, 4 December, 2009 *(10 minutes)*

Let $\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 1 & 0 & 3 \\ 0 & 1 & 2 & 1 \end{bmatrix}$. You may assume that the rank of $\mathbf{A}$ is 3.

1. Without any calculation, is $\mathbf{A}$ invertible? *[2]*
2. What is the nullity of $\mathbf{A}$? *[3]*

SOLUTION TO 1. No. $\mathbf{A}$ is not a square matrix, being 3×4, so it can't have an inverse. ■

SOLUTION TO 2. We are told that the rank of $\mathbf{A}$ is 3, so we can use the Rank-Nullity Law:

$$\text{nullity}(\mathbf{A}) = (\#\text{ of columns of }\mathbf{A}) - \text{rank}(\mathbf{A}) = 4 - 3 = 1 \qquad \blacksquare$$

Quiz #10. Friday, 11 December, 2009 *(10 minutes)*

1. Determine whether 4 is an eigenvalue of $\mathbf{A} = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{bmatrix}$. *[5]*

SOLUTION. Since $\mathbf{A}$ is an upper-triangular matrix, its eigenvalues are just the entries on its main diagonal, namely 1, 2, and 3. 4 isn't one of these, so it isn't an eigenvalue of $\mathbf{A}$. ■

NOTE. A slightly longer method would be to show that $\mathbf{A} - 4\mathbf{I}_3$ has nullity 0 (equivalently, has rank 3, is invertible, is non-singular, ...).