Mathematics 1350H – Linear algebra I: matrix algebra
TRENT UNIVERSITY, Fall 2009

Assignment #5
Due on Friday, 27 November, 2009.

Computing determinants using the Gauss-Jordan algorithm

Given a square matrix $\mathbf{A}$, we can compute a number called the *determinant* of $\mathbf{A}$, usually denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, that gives a lot of information about $\mathbf{A}$. For example, $|\mathbf{A}| \neq 0$ exactly when $\mathbf{A}^{-1}$ exists. Determinants turn up in various parts of mathematics besides linear algebra. For example, they are needed when changing coordinates when integrating in multivariate calculus.

A common problem with how determinants are usually defined is that computing them is a lot of work unless $\mathbf{A}$ is a pretty small matrix. (Heck, it's a pain even for 3×3 matrices with the usual definition ...) Here are some facts about determinants which let you compute the determinant of a matrix using the Gauss-Jordan algorithm. For large matrices, this is usually more efficient than using the standard definitions.

The determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If $i \neq j$ and you add any multiple of the jth row of $\mathbf{A}$ to the ith row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}|$.

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant, *i.e.* $|\mathbf{A}^T| = |\mathbf{A}|$.

(This collection of rules could be used as the definition of the determinant of a matrix.)

1. Why are rules *ii*–*iv* true for the columns of an $n \times n$ matrix as well as the rows? *[2]*

2. Use rules *i*–*v* and **1** to find the determinant of an $n \times n$ matrix $\mathbf{A}$ if:

a. $\mathbf{A}$ has a column or a row of zeros. *[2]*

b. $\mathbf{A}$ has two equal columns or two equal rows. *[1]*

c. $\mathbf{A}$ has rank less than n. *[1]*

4. Use the Gauss-Jordan method to put each of the matrices below in reduced row-echelon form and then apply what you have learned above to use this computation to find their determinants.

a. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$ *[1]* **b.** $\mathbf{B} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 3 & 3 & 15 \end{bmatrix}$ *[3]*