Trent University

MATH 1350H Test

3 November, 2008
Time: 50 minutes

Name: Solutions

Student Number: ____________________

Question	Mark
1	____
2	____
3	____
4	____
Total	____

Instructions

- *Show all your work.* Legibly!
- *If you have a question, ask it!*
- Use the back sides of the test sheets for rough work or extra space.
- You may use a calculator, and either (both sides of) one 8.5×11 aid sheet or a copy (annotated as you like) of *Formula for Success*.

1. Consider the points $(2, 0, 0)$, $(0, 2, 0)$, and $(0, 0, 2)$ in $\mathbb{R}^3$.

a. Find a parametric description of the line passing through the first two points. *[3]*

b. Find a linear equation describing the plane passing through all three points. *[4]*

c. Sketch the part of the plane in **b** that lies in the first octant. *[3]*

a. We'll use $(2, 0, 0)$ as the base point, and the vector from it to $(0, 2, 0)$ as the direction vector. The direction vector is therefore $\begin{bmatrix} 0-2 \\ 2-0 \\ 0-0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$, and the parametric representation of the line is $x = 2 - 2t$, $y = 0 + 2t = 2t$, and $z = 0 + 0t = 0$, where t is the parameter. ■

b. We need a vector normal to the plane and we'll get it from the cross-product of two vectors parallel to the plane. For these we'll use the vectors from $(2, 0, 0)$ to the other two given points. One of these we worked out in part **a**, and the other is $\begin{bmatrix} 0-2 \\ 0-0 \\ 2-0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}$.

Their cross-product is

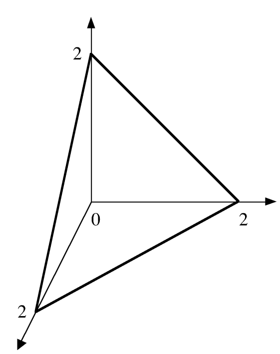
$$\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} \times \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 0 \\ -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -2 & 0 \\ -2 & 2 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -2 & 2 \\ -2 & 0 \end{vmatrix} \mathbf{k}$$

$$= (2 \cdot 2 - 0 \cdot 0)\mathbf{i} - ((-2) \cdot 2 - (-2) \cdot 0)\,\mathbf{j} + ((-2) \cdot 0 - 2 \cdot (-2))\,\mathbf{k}$$

$$= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix},$$

so the plane will have an equation of the form $4x + 4y + 4z = d$. To determine d, we simply plug the coordinates of one of the given points, say $(2, 0, 0)$, into the equation above and solve for d, $d = 4 \cdot 2 + 4 \cdot 0 + 4 \cdot 0 = 8$. Thus an equation of the plane is $4x + 4y + 4z = 8$. Those who like small numbers can divide both sides by 4 and use $x + y + z = 2$ instead. ■

c. We plot the intercepts of the plane, which conveniently happen to be the given points, and join them up:



■

2. Use the Gauss-Jordan method to find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix}$, if one exists. *[10]*

We row-reduce the "super-augmented" matrix:

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 4 & 0 & 1 & 0 \\ 1 & -1 & 10 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_2 - 2R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & -3 & 7 & -1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 - 2R_2 \\ \Longrightarrow \\ R_3 + 3R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 7 & 5 & -2 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right]$$

$$\begin{array}{c} R_1 - 7R_3 \\ R_2 + 2R_3 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 54 & -23 & -7 \\ 0 & 1 & 0 & -16 & 7 & 2 \\ 0 & 0 & 1 & -7 & 3 & 1 \end{array}\right]$$

It follows that the given matrix does have an inverse, and that

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 1 & -1 & 10 \end{bmatrix}^{-1} = \begin{bmatrix} 54 & -23 & -7 \\ -16 & 7 & 2 \\ -7 & 3 & 1 \end{bmatrix} . \quad \blacksquare$$

3. Do any *two* of parts **a**, **b**, **c**. *[10 = 2 × 5 each]*

a. Suppose $\mathbf{B}$ is an $n \times n$ matrix which is invertible and for which $\mathbf{B}^2 = \mathbf{B}$. Show that $\mathbf{B} = \mathbf{I}_n$, the $n \times n$ identity matrix.

b. Find the (shortest) distance from the point $P = (1, 0, 0)$ to the line ℓ given by the parametric equations $x = 1$, $y = 1 - 2t$, and $z = 1 + 3t$.

c. Can there be four planes in $\mathbb{R}^3$ which are each perpendicular to the other three? If so, give an example; if not, explain why not.

a. $\mathbf{B}^{-1}$ exists, so

$$\mathbf{B}\mathbf{B} = \mathbf{B}^2 = \mathbf{B} \quad\Longrightarrow\quad \mathbf{B}^{-1}\mathbf{B}\mathbf{B} = \mathbf{B}^{-1}\mathbf{B} \quad\Longrightarrow\quad \mathbf{I}_n\mathbf{B} = \mathbf{I}_n \quad\Longrightarrow\quad \mathbf{B} = \mathbf{I}_n\,,$$

as desired. ■

b. The direction vector of the line is $\mathbf{d} = \begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}$, and $t = 0$ gives us the point $(1, 1, 1)$ on the line. The vector joining the point $P = (1, 0, 0)$ to this point is $\mathbf{a} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and the distance from $P = (1, 0, 0)$ to the line is the part of $\mathbf{a}$ which is perpendicular to $\mathbf{d}$. We get this part by subtracting the projection of $\mathbf{a}$ onto $\mathbf{d}$ from $\mathbf{a}$:

$$\mathbf{a} - \mathrm{proj}_{\mathbf{d}}(\mathbf{a}) = \mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{d}}{\mathbf{d}\cdot\mathbf{d}}\mathbf{d} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}\cdot\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix}}\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{13}\begin{bmatrix} 0 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{15}{13} \\ \frac{10}{13} \end{bmatrix}$$

The distance between the point and the line is the length of this vector, namely:

$$\sqrt{0^2 + \left(\frac{15}{13}\right)^2 + \left(\frac{10}{13}\right)^2} = \sqrt{\frac{325}{169}} = \sqrt{\frac{25}{13}} = \frac{5}{\sqrt{13}}\,. ■$$

c. If there were four planes in $\mathbb{R}^3$ with each one perpendicular to the other three, we would have four vectors in $\mathbb{R}^3$ – namely the normal vectors of the planes – with each one perpendicular to the other three. This cannot happen, because the maximum number of vectors which are all perpendicular to each other in $\mathbb{R}^3$ is the dimension of $\mathbb{R}^3$, namely three. Hence there are no such planes. ■

4. Let $\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Do *one* of parts △ or □. *[10]*

△. Determine whether $\mathbf{d}$ is in $\text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ or not.

□. Determine whether $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are linearly independent or not.

△. $\mathbf{d}$ is in $\text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ if there exist x, y, and z such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{d}$. We try to solve the corresponding system of linear equations using the Gauss-Jordan method:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} R_2 \leftrightarrow R_4 \\ \Longrightarrow \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_4 - R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{a} + \mathbf{b} - 2\mathbf{c} = \mathbf{d}$, so $\mathbf{d}$ is in $\text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$. ■

△. The vectors are linearly independent if the only way to get $p\mathbf{a} + q\mathbf{b} + r\mathbf{c} + s\mathbf{d} = \mathbf{0}$ is to have $p = q = r = s = 0$. We try to solve the corresponding system of linear equations using the Gauss-Jordan method:

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} R_2 \leftrightarrow R_4 \\ \Longrightarrow \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_3 - R_1 \\ R_4 - R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{array}\right]$$

$$\begin{array}{c} \Longrightarrow \\ R_3 - R_2 \\ R_4 - R_2 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right] \begin{array}{c} \Longrightarrow \\ R_4 - R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that there are infinitely many solutions (you could let s be any real number and then solve for p, q, and r), and so the four vectors are not linearly independent, *i.e.* they are linearly dependent. ■

[Total = 40]