Mathematics 1350H – Linear algebra I: matrix algebra
Trent University, Fall 2008

Solutions to Assignment #5

Determinants the Gauss-Jordan way

Given a square matrix $\mathbf{A}$, we can compute a number called the *determinant* of $\mathbf{A}$, usually denoted by $|\mathbf{A}|$ or $\det(\mathbf{A})$, that gives a lot of information about $\mathbf{A}$. For example, $|\mathbf{A}| \neq 0$ exactly when $\mathbf{A}^{-1}$ exists. A common problem with how determinants are usually defined is that computing them is a lot of work unless $\mathbf{A}$ is a pretty small matrix. (Heck, it's a pain even for 3×3 matrices with the usual definition ...) Here are some facts about determinants which let you compute the determinant of a matrix using the Gauss-Jordan method:

The determinant of an $n \times n$ matrix $\mathbf{A}$ satisfies the following rules:

i. The identity matrix has determinant equal to 1, *i.e.* $|\mathbf{I}_n| = 1$.

ii. If you exchange the ith and jth row of $\mathbf{A}$ to get the matrix $\mathbf{B}$, then $|\mathbf{B}| = -|\mathbf{A}|$.

iii. If you multiply the ith row of $\mathbf{A}$ by a constant c to get the matrix $\mathbf{C}$, then $|\mathbf{C}| = c|\mathbf{A}|$.

iv. If you add a row vector $\mathbf{d}$ to the ith row of $\mathbf{A}$ to get the matrix $\mathbf{D}$, then $|\mathbf{D}| = |\mathbf{A}| + |\mathbf{A}_{i,\mathbf{d}}|$, where $\mathbf{A}_{i,\mathbf{d}}$ is the matrix $\mathbf{A}$ with its ith row replaced by $\mathbf{d}$.

v. Taking the transpose of $\mathbf{A}$ doesn't change the determinant. That is, $|\mathbf{A}^T| = |\mathbf{A}|$.

If you really wanted to, by the way, you could actually use this collection of rules as the definition of the determinant of a matrix.

1. Rules *ii* – *iv* are true for the columns of $\mathbf{A}$ as well as the rows. Why? *[2]*

Solution. Rule *v* is the reason[1]. Applying the operations mentioned in rules *ii* – *iv* to the columns of $\mathbf{A}$ corresponds to applying them to the rows of $\mathbf{A}^T$. Rule *v* tells us that $|\mathbf{B}| = |\mathbf{B}^T|$ for any matrix $\mathbf{B}$, so the effect on $|\mathbf{A}|$ of column operations on $\mathbf{A}$ is exactly the same as the effect on $|\mathbf{A}^T|$ of the corresponding row operations on $\mathbf{A}^T$. Hence rules *ii* – *iv* work for columns as well as rows. ■

2. Suppose we get the matrix $\mathbf{E}$ by adding a multiple of row i of $\mathbf{A}$ to row j of $\mathbf{A}$, leaving the other rows alone. Explain why $|\mathbf{E}| = |\mathbf{A}|$. *[2]*

Solution. Suppose we obtain $\mathbf{E}$ by adding c times row i of $\mathbf{A}$ to row j of $\mathbf{A}$. (That is, $\mathbf{A} \underset{R_j + cR_i}{\Longrightarrow} \mathbf{E}$.) Suppose $\mathbf{C}$ is the matrix $\mathbf{A}$ with row j replaced by c times row i, and $\mathbf{B}$ is the matrix $\mathbf{A}$ with row j replaced by row i. Then $|\mathbf{E}| = |\mathbf{A}| + |\mathbf{C}|$ (by rule *iv*) $= |\mathbf{A}| + c|\mathbf{B}|$ (by rule *iii*). Since $|\mathbf{B}| = 0$ by **3b**, it follows that $|\mathbf{E}| = |\mathbf{A}|$. (Note that the solution to **3b** does not rely on **2**, so we are not indulging in circular reasoning.) ■

[1] ... when rows are not in season!

3. Use rules i – v, as well as **1** and **2**, to compute $|\mathbf{A}|$ if:

a. $\mathbf{A}$ has a column or a row of zeros. *[1]*

Solution. Suppose $\mathbf{A}$ is an $n \times n$ matrix whose ith row, call it $\mathbf{r}_i$, is all zeros. Note that in this case $\mathbf{r}_i = 0\mathbf{r}_i$, so, by rule iii, $|\mathbf{A}| = 0|\mathbf{A}| = 0$.

If $\mathbf{A}$ has a column of zeros instead, then $\mathbf{A}^T$ must have a row of zeros, so $|\mathbf{A}| = |\mathbf{A}^T| = 0$, by the above and rule v (or by question **1**). ■

b. $\mathbf{A}$ has two equal columns or two equal rows. *[1]*

Solution. Suppose $\mathbf{A}$ is a matrix whose ith and jth rows are the same (with $i \neq j$, of course). Then $\mathbf{A} \underset{R_i \leftrightarrow R_i}{\Longrightarrow} \mathbf{A}$, so, by rule ii, $|\mathbf{A}| = -|\mathbf{A}|$. The only number which is equal to its own negative is 0, so it must be the case that $|\mathbf{A}| = 0$.

By problem **1**, it also follows that $|\mathbf{A}| = 0$ if $\mathbf{A}$ has two equal columns. ■

c. $\mathbf{A} = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$. *[1]*

Solution. Gauss-Jordan reduction, *hai!*

$$\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \overset{\frac{1}{3}R_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{4}{3} \\ 5 & 6 \end{bmatrix} \underset{R_2 - 5R_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & -\frac{2}{3} \end{bmatrix} \underset{-\frac{3}{2}R_2}{\Longrightarrow} \begin{bmatrix} 1 & \frac{4}{3} \\ 0 & 1 \end{bmatrix} \overset{R_1 - \frac{4}{3}R_2}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now we track how the determinant changed in the course of these operations until we reached the identity matrix:

$$|\mathbf{A}| \underset{(iii)}{\overset{\frac{1}{3}R_1}{\Longrightarrow}} \frac{1}{3}|\mathbf{A}| \underset{(\mathbf{2})}{\overset{R_2 - 5R_1}{\Longrightarrow}} \frac{1}{3}|\mathbf{A}| \underset{(iii)}{\overset{-\frac{3}{2}R_2}{\Longrightarrow}} -\frac{3}{2}\left(\frac{1}{3}|\mathbf{A}|\right) = -\frac{1}{2}|\mathbf{A}| \underset{(\mathbf{2})}{\overset{R_1 - \frac{4}{3}R_2}{\Longrightarrow}} -\frac{1}{2}|\mathbf{A}| = |\mathbf{I}_2| \underset{(i)}{=} 1$$

It follows that $|\mathbf{A}| = 1 \div \left(-\frac{1}{2}\right) = -2$. ■

d. $\mathbf{A} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. *[1]*

Solution. Following the same steps used in **3c** gives us:

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \overset{\frac{1}{p}R_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{q}{p} \\ r & s \end{bmatrix} \underset{R_2 - rR_1}{\Longrightarrow} \begin{bmatrix} 1 & \frac{q}{p} \\ 0 & \frac{ps-qr}{p} \end{bmatrix} \underset{\frac{p}{ps-qr}R_2}{\Longrightarrow} \begin{bmatrix} 1 & \frac{q}{p} \\ 0 & 1 \end{bmatrix} \overset{R_1 - \frac{p}{q}R_2}{\Longrightarrow} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(We are implicitly assuming here that none of p, q, and $ps - qr$ is 0.) Now we track how the determinant changed until we reached the identity matrix:

$$|\mathbf{A}| \underset{(iii)}{\overset{\frac{1}{p}R_1}{\Longrightarrow}} \frac{1}{p}|\mathbf{A}| \underset{(\mathbf{2})}{\overset{R_2 - rR_1}{\Longrightarrow}} \frac{1}{p}|\mathbf{A}| \underset{(iii)}{\overset{\frac{p}{ps-qr}R_2}{\Longrightarrow}} \frac{p}{ps-qr}\left(\frac{1}{p}|\mathbf{A}|\right) = \frac{1}{ps-qr}|\mathbf{A}|$$
$$\underset{(\mathbf{2})}{\overset{R_1 - \frac{p}{q}R_2}{\Longrightarrow}} \frac{1}{ps-qr}|\mathbf{A}| = |\mathbf{I}_2| \underset{(i)}{=} 1$$

It follows that $|\mathbf{A}| = ps - qr$, so long as none of p, q, and $ps - qr$ is 0. If $p \neq 0$ but $ps - qr = 0$, we reach a row of all zeros after two steps, and it is easy to see that in this case $|\mathbf{A}| = 0 = ps - qr$. If $p \neq 0$ and $ps - qr \neq 0$ but $q = 0$, we reach the identity matrix at the next-to-last step, and it is easy to see that in this case we still get $|\mathbf{A}| = ps - qr$. The cases where $p = 0$ are left to the reader. (If $r = 0$ too, it's easy; otherwise, swap rows first and apply the analysis above ...) ■

4. Use the Gauss-Jordan method to put the matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix}$ in reduced row-echelon form. Apply what you have learned above to use this computation to determine $|\mathbf{A}|$. *[2]*

Solution. First, we use the Gauss-Jordan method to put the matrix in reduced echelon form, one step at a time:

$$\begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 0 \end{bmatrix} \begin{matrix} R_1 \leftrightarrow R_2 \\ \Rightarrow \\ \end{matrix} \begin{bmatrix} 3 & 4 & 5 \\ 0 & 1 & 2 \\ 6 & 7 & 0 \end{bmatrix} \begin{matrix} \frac{1}{3}R_1 \\ \Rightarrow \\ \end{matrix} \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 2 \\ 6 & 7 & 0 \end{bmatrix} \begin{matrix} \\ \Rightarrow \\ R_3 - 6R_1 \end{matrix} \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 2 \\ 0 & -1 & -10 \end{bmatrix}$$

$$\begin{matrix} \\ \Rightarrow \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 1 & \frac{4}{3} & \frac{5}{3} \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{bmatrix} \begin{matrix} R_1 - \frac{4}{3}R_2 \\ \Rightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & -8 \end{bmatrix} \begin{matrix} \\ \Rightarrow \\ -\frac{1}{8}R_3 \end{matrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{matrix} R_1 + R_3 \\ \Rightarrow \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} \Rightarrow \\ R_2 - 2R_3 \\ \end{matrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Second, we track how the determinant changed in the course of these operations until we reached the identity matrix:

$$|\mathbf{A}| \underset{(ii)}{\overset{R_1 \leftrightarrow R_2}{\Longrightarrow}} -|\mathbf{A}| \underset{(iii)}{\overset{\frac{1}{3}R_1}{\Longrightarrow}} \left(\frac{1}{3}\right)(-|\mathbf{A}|) \underset{(\mathbf{2})}{\overset{R_3 - 6R_1}{\Longrightarrow}} -\frac{1}{3}|\mathbf{A}| \underset{(\mathbf{2})}{\overset{R_3 + R_2}{\Longrightarrow}} -\frac{1}{3}|\mathbf{A}| \underset{(\mathbf{2})}{\overset{R_1 - \frac{4}{3}R_2}{\Longrightarrow}} -\frac{1}{3}|\mathbf{A}|$$

$$\underset{(iii)}{\overset{-\frac{1}{8}R_3}{\Longrightarrow}} \left(-\frac{1}{8}\right)\left(-\frac{1}{3}|\mathbf{A}|\right) \underset{(\mathbf{2})}{\overset{R_1 + R_3}{\Longrightarrow}} \frac{1}{24}|\mathbf{A}| \underset{(\mathbf{2})}{\overset{R_2 - 2R_3}{\Longrightarrow}} \frac{1}{24}|\mathbf{A}| = |\mathbf{I}_3| \underset{(i)}{=} 1$$

Solving for $|\mathbf{A}|$ at the very end gives $|\mathbf{A}| = 24$. ■